ELSEVIER

Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

Theoretical Computer Science 305 (2003) 347–408

Theoretical Computer Science

www.elsevier.com/locate/tcs

Some points in formal topology

Giovanni Sambin

Dipartimento di Matematica Pura ed Applicata, Università di Padova, Italy

Dedicated to Silvia and Francesco
who have given me so much

What is this text?

My first paper on formal topology [32] was sent to the publisher fifteen years ago. It was the result of two years of intense work, in close collaboration with Per Martin-Löf. Since then, I never ceased to think and lecture about formal topology, and to discuss it with several people, mainly with Martin-Löf, Silvio Valentini (since 1991) and Thierry Coquand (since 1993), but later also with Giovanni Curi, Silvia Gebellato and Milly Maietti. I thank all of them, and all other people with whom formal topology has been shared. [1]

Since fifteen years ago, a lot has changed in formal topology, in technical and in conceptual understanding. Indeed, developing formal topology in a strictly constructive way, that is over an intuitionistic *and* predicative foundation (such as Martin-Löf's type theory, see below for a short introduction), has forced me to develop an alternative mathematical intuition and has helped me to reach a new global attitude towards the foundations of mathematics.

When I first gave a full course on formal topology, in Padua in 1990, I also first conceived the idea of writing a book on it. My manuscripts and typescripts have increased in number and length since then. However, in December 1995 it happened that I was able to materialize my new foundational attitude into a new fragment of mathematical thought, and I discovered what I called the basic picture: a clear, very simple structure underlying topology and consisting of symmetries and logical dualities. Mathematically, the basic picture is a natural generalization of topology, obtained by considering relations rather than functions as transformations. For the debate on foundations, it should be interesting that it is well visible only over an intuitionistic and predicative foundation, and this probably explains why it was not noticed before.

E-mail address: sambin@math.unipd.it (G. Sambin).

[1] I also thank Giovanni Curi, Per Martin-Löf and Peter Schuster for comments which have been useful in the revision of a first draft.

PII: S0304-3975(02)00704-1

The discovery of the basic picture caused a radical and extensive reformulation of formal topology itself, in particular by the introduction of a binary positivity predicate, the presence of co-induction and the generalization to nondistributive topology. This long paper contains the principles and the comments which I consider useful for the actual further development of this reformulation (and thus also of the book, whose writing has to be postponed). My ideal reader here is a colleague sincerely interested in constructive topology, including its conceptual motivations up to philosophical generalizations. So my aims are:

1. to give information on what has been done in formal topology since [32], and to give answers to some of the frequently asked questions about formal topology;
2. to show how the basic picture serves as the starting point for a new conception and a new technical development of constructive topology;
3. to verbalize my present understanding of how and why one should develop mathematics in a way which is different from usual ones.

These three items correspond to the three sections below, which can be read in any order. Readers less familiar with pointfree topology could look at the first parts of Section 2 and at Section 3.2.2 for some help.

The percentage of comments and personal opinions will be higher than what is to be expected according to a well-established tradition. I hope the readers (at least two, so that I can use "they" or "them" with no reference to gender) will forgive me and think that my effort of giving a general picture was worthwhile.

Some ideas about type theory

Martin-Löf's type theory [27] (henceforth type theory) aims at a constructive foundation of mathematics, which is alternative to usual axiomatic set theories such as ZFC and to categorical universes like topoi. I explain my reasons for choosing type theory as a foundation in part 3, and I show how this can be done in practice in Section 1.3.1. I recall here very briefly (and necessarily with imprecisions and gaps) some general facts about type theory, in particular those which make it different from ZFC or from topos theory. My aim is just to warn the readers, and possibly make them curious to learn more.

The name "type theory" is due to the fact that any entity goes together with its (logical) type. To know that A is a set means that we know by which rules all its elements are formed; these rules must be in front of us, hence in finite number, and cannot change with time.

If we know that A is a set, then we know the rules to produce its elements, and so clearly to say that a is an element of A means that a is produced by the rules; we write $a \in A$. In type theory one considers also objects which are not elements of a set, but rather belong to a collection, or logical type (or category in [27]). Two typical and important examples are the collection of subsets of a given set (see Section 1.3.1) and the collection of all sets (or propositions).

We can be in the position of knowing that $B(a)$ is a set, on the assumption that a is an element of a set A; this is called a family of sets indexed by A, and written $B(a)$ set $(a \in A)$. Similarly, we can be in the position of knowing that $b(a)$ is an element of a set B, on the assumption that $a \in A$; this is just our notion of function from A to B, and is written $b(a) \in B$ $(a \in A)$. More generally, if B is not a set but a family of sets depending on an index set A, $b(a) \in B(a)$ $(a \in A)$ means that we know $b(a)$ to be an element of $B(a)$ whenever $a \in A$. By abstracting on the variable a, we obtain the elements of a new set $(\Pi x \in A)B(x)$, called the direct product. If $c \in (\Pi x \in A)B(x)$ and $a \in A$, by applying c to a, we obtain an element of $B(a)$, which we here denote again by $c(a)$. In the special case in which B does not depend on A, the direct product is denoted by $A \to B$; if $c \in A \to B$ and $a \in A$, then $c(a) \in B$.

All other definitions of sets in type theory are given in a similar way. In particular, given a set A and a family $B(a)$ *set* $(a \in A)$, we can form the disjoint union $(\Sigma x \in A)B(x)$, whose canonical elements are pairs $\langle a, b \rangle$ with $a \in A$ and $b \in B(a)$. The special case in which B does not depend on A gives the cartesian product $A \times B$, the set of ordered pairs from A and B.

The standard formulation of type theory, as in [27] or [30], includes the interpretation of propositions as sets. So the judgement that P is a proposition obeys formally the same rules as the judgement that P is a set, reading $p \in P$ as p is a verification, or proof of P. In the proposition-as-set interpretation, $(\Pi x \in A)B(x)$ is identified with $(\forall x \in A)B(x)$ and $(\Sigma x \in A)B(x)$ with $(\exists x \in A)B(x)$, also in the sense that logical inference rules are just another reading of the rules for corresponding sets. When B does not depend on A, $(\forall x \in A)B(x)$ becomes implication $A \supset B$, here often written also as $A \to B$, and $(\Sigma x \in A)B(x)$ becomes conjunction $A \,\&\, B$. But this justification of logic is not necessary in formal topology; a treatment of logic which is distinct from the theory of sets is equally good, as long as it satisfies the rules of intuitionistic logic.

The only way to give a constructive proof that a certain property P holds for all the infinitely many elements of a set A, that is the only way to give meaning to a universal quantification over an infinite domain A, is by proving that P holds on an element only by virtue of its possible forms, which are only finitely many, that is one for each rule of A.

Type theory is quite different from an axiomatic theory of sets, such as ZF or ZFC. Using ZF as a foundation means assuming that ZF has a model, and this model is taken as a universe of all sets, in which mathematics is done using only the properties of sets as specified by the axioms. But such universe is considered as given, and thus there is no information on how the sets, and hence all other mathematical entities, are built up. When topos theory is assumed as a foundation, the universe is assumed to be a topos, and in this sense the situation is similar even if less properties are valid.

In type theory, the universe in which mathematics is done is built up in the same time as mathematics is built up. In practice, this means that whenever we use anything, we have total information about it, or total knowledge of what it is and by which ingredients it has been built up (while in ZF everything is reduced to only one ingredient, viz. sets, and only one relation, viz. membership). Note that this methodological request is not at all as unreasonable and difficult to fulfil as an education inside the ZF tradition might lead one to believe. On the contrary, it is very natural and simple when any entity must be constructed: indeed, it is enough to keep information about it in the same moment it is constructed. In type theory, the control of information is so strict that any proof of any statement is automatically also an algorithm, or computer program fulfilling that statement. This is the main source of strong interest in type theory by the computer science community.

In the classical approach, information is often lost in an irreversible way. An important example is the distinction between subset and characteristic function. Here a subset U of a set S is just a property, or propositional function $U(x)$ *prop* $(x \in S)$; we write as usual $U \subseteq S$. Note that a subset is never a set, because of the type difference.

If the collection of subsets is identified with the set of functions $S \to \{0,1\}$, namely characteristic functions, as it is done in ZF or in a boolean topos, then any subset U becomes decidable, in the sense that for each $x \in S$ one can decide whether $U(x)$ is true or not by calculating the characteristic function at x.

Here the collection of subsets of a set S is not a set, since by the well known phenomenon of diagonalization there is no way to give a finite stock of rules prescribing all the possible forms which all subsets should have (see [39]). This is what—in formal terms—blocks impredicativity, that is quantifications over subsets to produce a new proposition, or subset: simply, the subsets of a set do not form a set.

Note that it is not just a matter of names: one could call collections as "big sets" and sets as "tiny sets", or any other similar variation. The crucial point remains that a quantification over tiny sets does not produce a tiny set. This distinction is what makes the strict control of information at all possible. In particular, it allows to justify strong rules for disjoint unions, or for the existential quantifier. They permit to obtain, inside the formal system, two projection functions p, q which applied to a proof $c \in (\exists x \in A)B(x)$ produce an element $p(c) \in A$ and the proof that B holds on it, $q(c) \in B(p(c))$. This allows to justify also the axiom of choice: if $(\forall x \in A)(\exists y \in B)C(x,y)$ is provable, then also $(\exists f \in A \to B)(\forall x \in A)C(x, fx)$ is provable (see [27]).

This justification of the axiom of choice rests on the proposition-as-set interpretation. So, if logic is given independently of set theory, the axiom of choice is no longer justified. But the examples in which the axiom of choice is necessary seem, by experience, to be rare enough in formal topology that one can point them out case by case.

To be able to actually develop mathematics over type theory, one needs a number of tools, mainly notation and some auxiliary definitions, which are introduced in Section 1.3.1 below. I suggest the readers to read that section to be able to understand the notation which is used here from now on and which is by now standard in formal topology.

1. Some remarks, some results

In this section, I discuss the motivation of formal topology (Section 1.1), I justify the original definition and answer to some questions about it (Section 1.2), and briefly review the developments up to present (Section 1.3). The new approach to formal topology, which I have called the basic picture, will be treated in Section 2.

1.1. The point of formal topology

A topological space is classically defined (cf. e.g. [22,15]) as a pair $(X, \mathcal{O}X)$ where X is a set, whose elements $x, y, \ldots$ are called points, and $\mathcal{O}X$ is a family of subsets of X, which contains $\emptyset, X$ and is closed under finite intersections and arbitrary unions. The family $\mathcal{O}X$ is called a topology on the space X and the subsets in $\mathcal{O}X$ are said to be open.

The conditions on $\mathcal{O}X$ are written more precisely as:

$\mathcal{O}1$ $\emptyset, X \in \mathcal{O}X$,
$\mathcal{O}2$ for any $E, F, \subseteq X$, if $E, F \in \mathcal{O}X$ then $E \cap F \in \mathcal{O}X$,
$\mathcal{O}3$ for any family of subsets $\mathscr{F}$, if $\mathscr{F} \subseteq \mathcal{O}X$, then $\bigcup \mathscr{F} \in \mathcal{O}X$.

This formulation of the notion of topological space is unacceptable, as it stands, from a predicative point of view, since apparently a quantification not only over subsets, but over families of subsets (hence of the third order) is to be used. Though usually this is given meaning by conceiving the collection of subsets as a completed totality, we now see that actually no intrinsic impredicativity is involved, and that one can easily find a definition of topological space which is fully acceptable also predicatively.

A collection of subsets, and $\mathcal{O}X$ is one such, is most simply given in type theory as a set-indexed family, that is a function, which we call ext, from some set, which we call S, into $\mathscr{P}X$. In this way a quantification over open subsets—we cannot dispense with it in topology—can be reduced to a quantification over the set S.

However, one cannot expect ext to give all open subsets as values; the special case in which $\mathcal{O}X$ is the whole of $\mathscr{P}X$—the discrete topology—would require $\mathscr{P}X$ to become indexed by the set S, and this is not welcome in type theory.[2] Moreover, the expression of $\mathcal{O}3$ would still require an impredicative quantification.

These difficulties are solved by asking the family $\mathsf{ext}(a) \subseteq X$ $(a \in S)$ to be a base for the topology. Thus subsets $\mathsf{ext}(a)$ are called (basic) neighbourhoods, and open subsets are defined as arbitrary unions of neighbourhoods. That is, a subset D of X is open if and only if $D = \mathsf{ext}(U)$ for some $U \subseteq S$, where $\mathsf{ext}(U) \equiv \bigcup_{b \varepsilon U} \mathsf{ext}(b)$. Without loss of generality,[3] we may assume that the usual conditions on bases are satisfied in the sense that S is provided with a binary operation $\cdot$ and with a distinguished element 1 such that

$$\mathsf{ext}(1) = X \quad \text{and} \quad \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{ext}(a \cdot b)$$

The resulting structure is called a *concrete space* (in [32, example 2.1]), or a concrete presentation of a topological space.

From our constructive point of view, this definition is certainly acceptable, but not sufficient to develop topology. One has to add two further notions, that of formal topology and that of formal point, and hence also that of formal space, as the collection of all formal points. In fact, in many interesting examples, the collection X of points of a classical topological space is not given directly as a set, in the constructive sense. And this may happen also when basic neighbourhoods of X can be given as a family ext indexed on a set S. The reason for this is that an infinite amount of information,

[2] By adapting to type theory the well-known argument due to Diaconescu, it is shown in [24] that this would bring to classical logic.

[3] This is actually not completely true: for 10 years it has prevented me from seeing an important generalization, see Section 2.

which means infinitely many basic neighbourhoods, may be necessary to determine a point. The idea is then to consider elements $a, b, c \ldots$ of S as formal neighbourhoods, and hence subsets $U, V, \ldots$ of S as formal substitutes of open subsets. One has to define, however, when two subsets of S are topologically equivalent, that is when they produce the same open subset. This leads to the definition of formal topology, which thus is a specific structure on the set of formal neighbourhoods. Then an infinite amount of information can be given by a subset α of S, and when α has some properties which make it formally similar to a point, it will be called a formal point.

The method to obtain the definition of a formal notion, those of formal topology and formal point to begin with, is always the same, and it can be described as formed by three steps:

1. Study the notion to be defined in the presentable case, in which both a set of points X and a set of formal neighbourhoods S are present. This allows to choose some new primitives to be added to the formal side, in view of step 2.
2. Analyse the structure induced on the formal side, and write down all those properties of the primitives on S which can be expressed without mentioning the points of X. Of course, the best choice of primitives is that which allows to describe the original concrete notion in the best possible way.
3. Abandon points altogether, and retain those properties of formal primitives as an axiomatic definition.

We apply this method first of all to obtain the definition of formal topology itself. In the concretely presentable case, two subsets U, V of S correspond to the same open subset of X when $\mathsf{ext}\, V = \mathsf{ext}\, U$. To express this in pointfree terms, it is enough to express $\mathsf{ext}\, V \subseteq \mathsf{ext}\, U$, and this in turn, by the definition $\mathsf{ext}\, V = \bigcup_{a\,\varepsilon\, V} \mathsf{ext}\, a$, reduces to $(\forall a\, \varepsilon\, V)(\mathsf{ext}\, a \subseteq \mathsf{ext}\, U)$. So we add an infinitary relation $a \lhd U$ as primitive, with the idea that it corresponds to $\mathsf{ext}\, a \subseteq \mathsf{ext}\, U$; using it one can define $V \lhd U \equiv (\forall a\, \varepsilon\, V)(a \lhd U)$, which then corresponds to $\mathsf{ext}\, V \subseteq \mathsf{ext}\, U$, and finally $V =_{\lhd} U \equiv V \lhd U \,\&\, U \lhd V$ will correspond to $\mathsf{ext}\, V = \mathsf{ext}\, U$.

The distinguished element 1 and the operation $\cdot$ are also kept, and the idea is that $\mathsf{ext} 1 = X$ and that $\mathsf{ext}(a \cdot b) = \mathsf{ext}\, a \cap \mathsf{ext}\, b$. Finally, we also add a unary predicate $Pos(a)$ *prop* $(a \in S)$, whose meaning in the concrete case is that $\mathsf{ext}\, a$ is inhabited; in fact, this is constructively not reducible to $\mathsf{ext}\, a \neq \emptyset$. The result of applying now steps 2 and 3 of the method above is the definition of formal topology given in next section.

The method by which we reached the definition says that any concrete space gives a formal topology, which is then called (*concretely*) *presentable*. But note that not all formal topologies are presentable;[4] if it were so, their introduction would be much less motivated. A detailed discussion of the motivations for the introduction of the notion of formal topology is in Section 3.2.2.

[4] An example of non-presentable formal topology is given in [9], but simpler, finite examples can be built up.

1.2. A formal topology is...

The first result of the method described in the previous section is the definition of formal topology itself. The following is a minor variant (but equivalent from many aspects)[5] of the original in [32]:

Definition 1. A formal topology $\mathcal{S}$ consists of:

a set S,
a distinguished element 1 and a binary operation $\cdot$ on S,
a relation $\lhd$ between elements and subsets of S, called (formal) cover, which for arbitrary $a, b \in S$, $U, V \subseteq S$ satisfies:

reflexivity $\dfrac{a\,\varepsilon\,U}{a \lhd U}$,

transitivity $\dfrac{a \lhd U \qquad U \lhd V}{a \lhd V}$ where $U \lhd V \equiv (\forall b\,\varepsilon\,U) b \lhd V$,

$\cdot$-Left $\dfrac{a \lhd U}{a \cdot b \lhd U} \qquad \dfrac{a \lhd U}{b \cdot a \lhd U}$,

$\cdot$-Right $\dfrac{a \lhd U \qquad a \lhd V}{a \lhd U \cdot V}$ where $U \cdot V \equiv \{a \cdot b\colon\ a\,\varepsilon\,U, b\,\varepsilon\,V\}$,

top $a \lhd 1$,

a predicate $\mathsf{Pos}(a)$ on S, called positivity predicate, which for arbitrary $a \in S$, $U \subseteq S$ satisfies:

monotonicity $\dfrac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists b\,\varepsilon\,U)\mathsf{Pos}(b)}$,

positivity $\dfrac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$.

A frequently asked question about formal topology is whether quantification over subsets is really avoided; the claim is that the very definition of formal topology involves a quantification over subsets. The crucial point is of course the use of "for arbitrary U", especially in a formalistic reading of the definition. The answer is that we use subset variables as arguments of (higher order) functions, that is we do *not* use them to build up new propositions (that is, new subsets) of the form $\forall U \ldots$, but keep always the quantification at the meta-level (formally: subset variables remain free). So the definition with all details should read:

Definition 2. A formal topology $\mathcal{S}$ consists of:

a set S, which is determined by specifying its introduction and elimination rules;

[5] The aim of this variant is to avoid problems connected with equality; usually $(S, \cdot, 1)$ is assumed to be a monoid or a semilattice, which is expressible only using equality of S. Equivalence holds in the sense that putting $a =_{\lhd} b \equiv (a \lhd \{b\}\ \&\ b \lhd \{a\})$, one can show that $(S, \cdot, 1, =_{\lhd})$ is a semilattice.

a distinguished element 1 and a binary operation $\cdot$ on S, that is $a\cdot b\in S$ $(a\in S, b\in S)$;

a relation $\lhd$ between elements and subsets of S, that is $a\lhd U$ *prop* $(a\in S, U\subseteq S)$ (which will be defined as usual for any proposition by furnishing introduction and elimination rules, either directly or indirectly by means of an expression like a logical formula, of which we already know that it produces propositions), and six functions *refl*, *trans*, l_1, l_2, t and r of the convenient types which satisfy:

reflexivity $\quad refl(a,U)\in U(a)\to a\lhd U \ (a\in S, U\subseteq S)$,

transitivity $\quad trans(a,U,V)\in a\lhd U \,\&\, U\lhd V\to a\lhd V \ (a\in S, U, V\subseteq S)$,

$\cdot$-Left $\quad l_1(a,b,U,V)\in a\lhd U\to a\cdot b\lhd V \ (a\in S, U\subseteq S)$
$\quad l_2(a,b,U,V)\in a\lhd U\to b\cdot a\lhd V \ (a\in S, U\subseteq S)$,

$\cdot$-Right $\quad r(a,U,V)\in a\lhd U \,\&\, a\lhd V\to a\lhd U\cdot V \ (a\in S, U, V\subseteq S)$,

top $\quad t(a)\in a\lhd 1$;

a predicate $\mathsf{Pos}(a)$ on S, that is $\mathsf{Pos}(a)$ *prop* $(a\in S)$, and two functions m and p which satisfy:

monotonicity $\quad m(a,U)\in \mathsf{Pos}(a)\,\&\, a\lhd U\to(\exists b\,\varepsilon\, U)\mathsf{Pos}(b) \ (a\in S, U\subseteq S)$,

positivity $\quad p(a,U)\in(\mathsf{Pos}(a)\to a\lhd U)\to a\lhd U \ (a\in S, U\subseteq S)$.

This formalistic definition (with proof-terms spelled out to please a computer language) has absolutely no quantification over subsets. I never wrote it explicitly before, because I assumed it was understood.[6] The notation with hidden proof-terms is more suitable to human mathematicians. Keeping explicit track of all the proof-terms, that is of computational content, would impede a more abstract understanding, or at least would make it much harder.[7] In any specific example, of course, one has to produce, at least in principle, *all* the required information, so including the functions in variant 1.2, simply to be sure that one has actually *given* an example of formal topology.

The apparent quantification over subsets needed in the definition of formal topology is of the same kind as the quantification over propositions A, B which is needed to understand a simple inference rule such as

$$\frac{A}{A\vee B}.$$

In fact, one understands here that the rule applies to any propositions A and B, but nobody has ever questioned whether a second-order quantification is here involved, since it is clear that the quantification involved remains at the metalevel.

However, one must be *extremely* careful on this topic, since not all quantifications at the metalevel are equally innocent. Let me first recall one aspect of the intuitionistic

[6] Definition 2 was given for the first time explicitly in my talk at TYPES'98, Kloster Irsee, together with the comments given in this section.

[7] To be pedantic, this is an example of the forget-restore principle (see Section 3.1.3): one should make sure that hiding the proof-terms of all the propositions does not prevent us from obtaining them back when wished. This is possible because all proofs will be intuitionistic, and thus preserve proof-terms.

meaning of quantifiers. The meaning of a statement of the form $(\forall x \in S)A(x)$ in intuitionistic terms is that we have a method proving $A(a)$ for every $a \in S$. So the meaning of $(\forall x \in S)(\exists y \in S)A(x, y)$ is that we have a method which applies to any $a \in S$ and produces a proof of $(\exists y \in S)A(a, y)$, that is an element c, depending on a, such that $A(a, c)$ holds.

It seems to me that there is no other way to give constructive meaning to a universal-existential statement, also when quantifiers are meant to be kept at the metalevel. So I am able to grasp that "for every $U \subseteq S$, there exists b such that $A(U, b)$" holds only when I *have* a function F such that $A(U, F(U))$ holds for every $U \subseteq S$. One can debate whether this function should always be expressible *within* the language. But assuming that the meaning of "for every U there exists b" is always predicatively clear (which is implicit when such a combination of quantifiers is used to define an object, like a subset) amounts to assuming that the function F can be obtained always, and that it is expressible in the language, which means that a second-order axiom of choice of the kind $\forall U \exists b A(U, b) \rightarrow \exists F \forall U A(U, F(U))$ must hold. But then this brings us immediately to classical logic (see [24] for the precise statement and proof of this fact).

This is an example of a "powerful" principle which actually destroys the quality of information or equivalently, at least in my own case, which destroys the possibility of an intuitive grasping (see Section 3.1.3). A consequence is that in formal topology one will always find directly the function F, and never the combination "for every U, there exists b" (or "for every U, there exists W") to which it gives meaning (see for instance the case of the definition of $U \downarrow V$ in Section 2.1).

Another critique to the definition of formal topology is that… there are too many different definitions. I would just like to recall that even what now looks as the most stable definition of (usual) topology, namely that of topological space, is actually the result of a long historical process, which stabilized relatively recently. One advantage of the variant given above is explained in footnote 5. Two further variants will be introduced in Sections 2.1 and 2.4, together with some good reasons to do it.

1.3. A summary of developments

Starting from the definition of formal topology, the paper [32] contains other basic definitions (formal open, formal point, formal space, continuous function,…) some technical tools (connection with closure operators,…), and several examples (Scott domains, Stone representations, choice sequences, real numbers,…); it also begins the use of inductive methods in topology, which is peculiar to formal topology. I have grouped the subsequent developments under seven headings, as follows. I do not insist on details, or any kind of information like dates and credits, whenever there is a good source for this. As often happens, the official dates of publication do not correspond to the time things were first discovered.

1.3.1. A toolbox to do mathematics in type theory

In the preface of [32] it was said that a subset is just a propositional function, but the adherence to this principle in the subsequent development was not based on a rigorous and full formalization in type theory. A satisfactory understanding, both in practical and in formal terms, came only a few years later (see [39]), and it mostly confirmed the first intuitions. Now I believe that a comprehensive book on formal topology should contain a substantial chapter with a detailed explanation of how mathematics can be developed in practice using type theory as a foundation.

Using type theory to do mathematics is not so different from using ZF, to the extent that in both cases one needs a set of tools (definitions, macros, notation, abbreviations, etc.), or toolbox, to avoid clumsy formalities. The tuning up of such a toolbox for ZF has required some effort (think for example of the time passed between Zermelo's axioms and the formal treatment of ordered pairs). The analogue process for type theory has started only relatively recently.

The main observation is that in the practice of mathematics one is often not interested in all the amount of information which type theory is able to preserve (this is also true for ZF, but to a lesser extent). The general idea is that, however, one should not forget information in an irreversible way: ideal is the situation when any piece of information which does not appear explicitly can be restored at will, maybe by passing through the metalanguage (see [39] and Section 3.1.3 here).

Once the toolbox which is necessary to develop a certain field is built up, doing mathematics in type theory is just doing mathematics: all the boring details of actual formalization in type theory are taken care of once and for all at the time the toolbox is implemented. An essential aspect, and a definite advantage, of this approach is that the basic (type) theory is kept fixed, while the toolbox is expanded, and this allows stratification of knowledge and increase of confidence.

At present, a complete toolbox for subsets [8] has been developed, in [39]. I here repeat—with mild variations—the bare minimum to be able to read this text. As mentioned in the preface, a *subset* U of a set S is just a propositional function, that is a proposition $U(x)$ for each $x \in S$. We write $U \subseteq S$ as usual. To be able to use spatial intuition, and to keep closer to mathematical practice, we want to introduce the notion of element a of a subset U, which we write as $a \,\varepsilon_S\, U$ (forgetting the subscript whenever possible). The idea is that $a \,\varepsilon_S\, U$ is equivalent to $a \in S$ and $U(a)$ *true*. This is equivalently expressed by the conditions that

(i) $(\forall x \in S)\ (x \,\varepsilon_S\, U \leftrightarrow U(x))$ must be true, that is, when a is known to be in S, $a \,\varepsilon_S\, U$ is logically equivalent to $U(a)$, and

(ii) if $a \,\varepsilon_S\, U$ *true*, then $a \in S$, that is, the proposition $a \,\varepsilon_S\, U$ *true* keeps the information that $a \in S$ so that there is no need to recover $a \in S$ from $a \,\varepsilon_S U$ by inspecting the proof.

These two conditions are satisfied for instance by setting $a \,\varepsilon_S\, U \equiv Id(S, a, a) \,\&\, U(a)$, where Id is intensional propositional equality (see [30]), but other solutions are possible and equally good for the purposes we are now discussing. In fact, the theory of subsets is developed on the basis of the above two conditions only, so that one can ignore how they are actually implemented in type theory. [9]

---

[8] We also know how to deal with quotients and with quantifications over finite subsets.

[9] Another consequence is of course that the job of implementation of the whole theory of subsets reduces to the implementation only of the two conditions, see Section 3.2.1.

The idea underlying ε can also be expressed as the wish to keep the notation $\{x \in S\colon\ U(x)\}$, in the usual sense that $a\,\varepsilon\,\{x \in S\colon\ U(x)\}$ if and only if $a \in S$ and $U(a)$ *true*. Thus $\{x \in S\colon\ U(x)\}$ becomes just a "shorthand" for U. Another relevant fact is that any subset of S is equal (in the sense below) to the image of a set I along a function $f : I \to S$, which is defined by $f[I] \equiv \{x \in S\colon\ (\exists i \in I)x = f(i)\}$. So one could say that the distance between a subset and a set is only one function!

Equality between subsets is extensional. That is, for any $U, V \subseteq S$ inclusion is defined by $U \subseteq V \equiv (\forall x \in S)\ (x\,\varepsilon\,U \to x\,\varepsilon\,V)$ and then equality by $U = V \equiv U \subseteq V \,\&\, V \subseteq U$.

The common pattern behind the definition of an operation on subsets is that it is simply the abstraction, at the level of propositional functions, of a logical constant, which acts at the level of propositions. For example, intersection $\cap$ is the abstraction of conjunction $\&$, that is $U \cap V \equiv \{x \in S\colon\ U(x) \,\&\, V(x)\}$. This is not sheer manipulation of symbols, but making the link between visual intuition and logic explicit; for instance, the statement: $a\,\varepsilon\,U \cap V$ if and only if $a\,\varepsilon\,U$ and $a\,\varepsilon\,V$, without notation with ε would be just the definitional equation $(U \cap V)(x) \equiv U(x) \,\&\, V(x)$, and thus its intuitive value would be partly lost. Similarly, $U \cup V \equiv \{x\colon\ U(x) \vee V(x)\}$ and $-U \equiv \{x :\ \neg U(x)\}$ (remind that, because of intuitionistic logic, one cannot expect $S = U \cup -U$ to hold in general).

A family of subsets of S indexed by a set I, written $U_i \subseteq S(i \in I)$, is just the same thing (but not the same intuition!) as a *binary relation* between I and S, that is a propositional function with two arguments $U(i,x)$ *prop* $(i \in I, x \in S)$. Clearly, equality of families of subsets is extensional, that is $U_i \subseteq S(i \in I)$ and $V_i \subseteq S(i \in I)$ are said to be equal if $U_i = V_i$ for each $i \in I$. This yields also that two relations are said to be equal if they hold on the same arguments. All definitions and results dealing with binary relations must be understood up to this equality.

Following the pattern mentioned above, the union of a set-indexed family of subsets $U_i \subseteq S(i \in I)$ is defined by abstracting the existential quantifier: $x\,\varepsilon \bigcup_{i \in I} U_i \equiv (\exists i \in I)(x\,\varepsilon\,U_i)$. Similarly for intersection and universal quantifier.

The collection of subsets of a set S, equipped with extensional equality, is called the power of S and is denoted by $\mathscr{P}S$. The above operations give to it the structure of a frame (or complete Heyting algebra, or locale). A rigorous proof of this is obtained by noting that, since an operation on subsets is the abstraction of a logical constant, any informal argument about operations on subsets is always supported by a formal logical deduction about the corresponding logical constant. For example, the properties of inclusion with respect to union and intersection, namely

(a) $\bigcup_{i \in I} U_i \subseteq V$ if and only if for all $i \in I$, $U_i \subseteq V$
(b) $V \subseteq \bigcap_{i \in I} U_i$ if and only if for all $i \in I$, $V \subseteq U_i$

are obtained immediately by a shift of quantifiers.

Since the writing of [39], an important improvement in notation has taken place. Since the quantifier $\exists$ is primitive, and not definable by means of $\forall$, it is convenient to introduce a notation for the notion which is dual to that of inclusion. That is, for any $U, V \subseteq S$ we put

$$U \between V \equiv (\exists a \in S)(a\,\varepsilon\,U \,\&\, a\,\varepsilon\,V)$$

and we read “U meets V”. Note that $U \between V$ is intuitionistically much stronger than $U \cap V \neq \emptyset$. The property corresponding to (a) and (b) above is:

(c) $\bigcup_{i \in I} U_i \between V$ iff there exists $i \in I$ such that $U_i \between V$.

We will see in Section 2 that the notation $\between$ is very useful for the expression of some fundamental mathematical properties.

One can define singletons by $\{a\} \equiv \{x \in S \mid a = x\}$ (so that by logic $a \, \varepsilon \, U$ iff $\{a\} \subseteq U$ iff $\{a\} \between U$), finite subsets by $\{a_0, \ldots, a_{n-1}\} \equiv \{x \in S \mid x = a_0 \vee \cdots \vee x = a_{n-1}\}$ whenever $a_0, \ldots, a_{n-1} \in S$ (keeping in mind that several intuitionistically non equivalent notions are reasonable), etc.; I refer to [39] for details on these and other tools, like quantifiers relative to a subset, subset-indexed family of subsets, etc.

Other tools are still to be tuned up and tested. Note that this is not a routine task. We know for instance that requiring some among the common properties of powersets and of quotient sets, or that the subsets of a set, or even the finite ones, form a set, would bring us to classical logic (see [24,23]); this would mean destroying all the efforts of preserving constructivity. To develop formal topology such properties are not essential; for instance, quantification over finite subsets of a set (or of a subset) is reduced to quantification over a set of lists.

1.3.2. Predicative completeness proofs

An important insight about intuitionistic logic, which goes back to the 1930s, is that its propositions (or formulae) can be interpreted mathematically as the open subsets of a topological space. As shown in [33], also formal topology provides with a complete semantics, by interpreting formulae as formal open subsets (and by suppressing the predicate Pos). Since the notion of formal topology is fully predicative, the result is a proof of completeness of topological semantics which is also fully predicative. As in the original proof by Henkin, the key step is the construction of a generic model from the syntax itself; in our case, a suitable cover on the set of formulae must be introduced. Two such covers are studied in detail in [9], where it is shown that formal points over one of them are exactly the same thing as Henkin sets. This gives a precise form to the idea that points correspond to models [16]; for some other comments and references, see the introduction of [9].

The completeness proof in [33] is actually given in a modular way for a variety of logics, which all are extensions of intuitionistic linear logic. To this aim, the notion of cover is generalized to that of precover, in which the two assumptions on $\cdot$, namely $\cdot$-Left and $\cdot$-Right, are replaced by the single one:

$$\text{stability} \qquad \frac{a \lhd U \qquad b \lhd V}{a \cdot b \lhd U \cdot V}.$$

or its equivalent

$$\text{localization} \qquad \frac{a \lhd U}{a \cdot b \lhd U \cdot b} \quad \text{where } U \cdot b \equiv U \cdot \{b\}.$$

A pretopology is a commutative monoid equipped with a precover. A cover becomes exactly the same as a precover satisfying the conditions corresponding to the structural rules of weakening and contraction, which can be seen to be $a \cdot b \lhd a$ and $a \lhd a \cdot a$, respectively (or some other equivalents). On the other hand, pretopologies in which the double negation law is valid turn out to coincide literally with phase spaces, that is the semantics of linear logic given by Jean-Yves Girard in [20].

1.3.3. *Predicative presentation of frames*

An infinitary relation $\lhd$ satisfying only the properties of reflexivity and transitivity, as in the definition of covers, is called an infinitary preorder. It was discovered long ago (see [32]) that infinitary preorders on a set S correspond biunivocally to closure operators on S (that is, functions $\mathscr{C} : \mathscr{P}S \to \mathscr{P}S$ such that $U \subseteq \mathscr{C}U$, $U \subseteq V \to \mathscr{C}U \subseteq \mathscr{C}V$ and $\mathscr{C}\mathscr{C}U \subseteq \mathscr{C}U$). In fact, by setting $\mathscr{A}U \equiv \{a \in S : a \lhd U\}$ one has that $a \lhd U$ is the same as $a \,\varepsilon\, \mathscr{A}U$, so that reflexivity can be rewritten as $U \subseteq \mathscr{A}U$ and transitivity as $V \subseteq \mathscr{A}U \to \mathscr{A}V \subseteq \mathscr{A}U$; one can then easily check that these two conditions on $\mathscr{A}$ are equivalent to those in the definition of closure operator. Moreover, it is well known that closure operators correspond to complete lattices (given a closure operator $\mathscr{A}$, the collection of saturated subsets $Sat(\mathscr{A}) \equiv \{U \subseteq S\colon\ U = \mathscr{A}U\}$ is a complete lattice, in which meet is given by intersection and join by the saturation of union, and conversely, given a complete lattice, putting $a \,\varepsilon\, \mathscr{A}U \equiv a \leqslant \bigvee U$ gives a closure operator).

Building on these remarks, one can obtain a modular presentation of sup-lattices (that is, lattices with arbitrary joins—and hence also meets—but in which only joins are preserved by morphisms), quantales and frames by generators and relations. The sup-lattice freely generated by a set S of generators is just $\mathscr{P}S$. So the idea is to describe the ordering of *any* sup-lattice generated by S by adding conditions, or axioms $R(a, U)$, to be satisfied if $a \leqslant \bigvee_{b \,\varepsilon\, U} b$. The main result (which generalizes a similar result in [21]) is that the least infinitary preorder $\lhd_R$ containing R gives exactly the free sup-lattice satisfying the axioms given by R. The same result for quantales and frames is obtained in a modular way, by adding suitable extra conditions.

This line of research was begun very early, see [2], and several earlier versions of the final paper [3] circulated privately. In fact, it took a long time to understand properly how it is possible to generate $\lhd_R$ above in a predicative way, and for which R this is possible (see [10]). One must be very careful here: when one says that formal topologies (without Pos) form a category which is equivalent to that of frames, one must realize that the proof cannot be predicative, unless one previously restricts to a predicative definition of frames. The point here is that a predicative notion of frame… is nothing but the notion of formal topology.

1.3.4. *Domain theory as a branch of formal topology*

For any formal topology $\mathscr{S}$, the collection of its formal points $Pt(\mathscr{S})$ (see Section 2.4 for a definition) is said to be a *formal space*. This is a genetic characterization of formal spaces. In general, an axiomatic definition is not available; one can only define as usual the specialization ordering on formal points α, β by setting $\alpha \leqslant \beta \equiv \beta \subseteq \alpha$

(α is less than β if it is more informative, i.e. contains more elements of S) and observe that $Pt(\mathcal{S})$ thus becomes a complete partial order. But if we restrict our attention to the class of unary formal topologies, which are those in which the cover is unary, or 1-compact,

$$a \lhd U \quad \text{iff } \mathsf{Pos}(a) \rightarrow (\exists b\, \varepsilon\, U)(a \lhd \{b\}),$$

(It is understood that this equivalence can hold constructively only if we have a function $F(a,U)\,\varepsilon\, U \quad (a \in S, U \subseteq S)$ such that $a \lhd F(a,u)$ whenever $a \lhd U$ (see the discussion at the end of Section 1.2).) then the associated class of formal spaces admits of an axiomatization, and actually a well-known one, since it turns out to be exactly the class of Scott domains (the link with Scott domains was present from the beginning, see [32], Section 7, but it was spelled out only later in [40]). In fact, a unary cover is intuitively one in which no two neighbourhoods do cooperate to produce coverings. So one can see that in any unary $\mathcal{S}$ all subsets of the form $\uparrow a \equiv \{b : a \lhd b\}$, for any positive a, are formal points of $\mathcal{S}$, and all formal points are obtained by forming unions of these. In other words, positive elements of $\mathcal{S}$ correspond to compact elements of the Scott domain $Pt(\mathcal{S})$. Then one can read both Scott's definition of information systems [41] and the even simpler definition of information bases in [40] either as an axiomatization of the structure of compact elements of a domain or as a simplified characterization of unary formal topologies. This is to say that the category of unary formal topologies, that of information systems, and that of information bases are mutually equivalent, and Scott domains are obtained by applying the functor Pt bringing to formal spaces. So the definitions of domain theory become special cases of notions having a general topological meaning, and in the end this has produced a simplified approach to the theory of domains, which moreover is fully predicative. For instance, it has been proved predicatively, by Valentini [47], that the category of information bases is cartesian closed.

The connection between formal topology and domain theory is clear also in the approach to formal topology via the basic picture, which is described in Section 2 below. A curious fact is that, while the categories of (arbitrary) formal topologies, in the old and in the new sense, are equivalent, this is no longer true for unary formal topologies. So unary formal topologies, in the new sense, are equivalent to algebraic domains, and the extra condition characterizing $Pt(\mathcal{S})$ as a Scott domain is *not* independent of the way $\mathcal{S}$ is given (see [34]). The next natural step is to extend the connection with domain theory by finding predicative definitions of the way-below relation and of continuous domains; a common expectation is that the right idea should be that of bases with the interpolation property (see [1]).

A nice topic for research is to reveal which of the results for unary formal topologies extend to the case of finitary (or Stone) formal topologies, that is those in which the cover satisfies

$$a \lhd U \quad \text{iff there is a finite subset } K \text{ of } U \text{ such that } a \lhd K.$$

(Here too one understands the presence of a function $F(a,U)$ such that $F(a,U)$ is a finite subset of U and $a \lhd U \rightarrow a \lhd F(a,U)$.) In particular, it is still unknown to me whether it is possible to find an axiomatization of formal spaces corresponding to finitary formal topologies.

Also quantitative domain theory can be dealt with predicatively. Curi has shown in [11,12] that the notion of cpo's with weight and distance of [49] can be generalized to the notion of a formal topology with weight and distance.

1.3.5. Inductive generation of formal topologies and proof-theoretic methods

A formal topology, one could say, is just a way to present a frame (the frame $Sat(\mathscr{A})$) by generators (the set S) and relations (the cover $\lhd$, or equivalently the closure operator $\mathscr{A}$). The choices taken when defining formal topologies are actually linked with the choice for predicative methods. But whatever the reason is, the introduction of formal topologies has opened the way to the use of inductive methods in topology. Actually, all the axioms or conditions are preferably written in the form of inference rules exactly for the purpose of applying proof-theoretic methods or ideas. This appears as a conceptual novelty in the field of topology, and gives to formal topology its distinctive character: formal topology, which happened to begin as a theory of locales developed over a different foundation (namely, type theory rather than topos theory), has later developed a specific identity also from a strictly mathematical point of view. One typical result in this sense is the normal form theorem for covers on real numbers, and the problem it leads to (see Section 1.3.6 below). Another one is that the finitary content of a formal cover generated by axioms Σ is just the cover generated by the finitary part of Σ, that is, by those axioms of Σ in which only finite subsets are involved.

The importance of the inductive generation of formal topologies is clear, for a predicative approach, when one observes that, for instance, the product of two formal topologies cannot be defined predicatively, unless they are inductively generated (see [10]). This has raised the question whether one should restrict one's attention and add the requirement of inductive generation to the definition of formal topology itself; this is discussed in Section 2.6.

Any other information about the inductive generation of formal topologies can be found in [10]; in particular, the readers will discover there that allmost all the examples of formal topologies which can be found "in nature" do fall under the scope of the theorem on inductive generation. This gives a solid argument in favour of formal topology, since it automatically means that all those examples can be formalized into a computer language.

1.3.6. The continuum as a formal space

In [32] it was suggested that the continuum could be presented via formal topology essentially as in [21]. This idea was later worked out by my student Daniele Soravia in [45], where also the beginning of real analysis is developed (all this appeared subsequently in [28]). The main idea is that a real number is a formal point on a suitable formal topology where basic neighbourhoods are pairs of rational numbers, (p, q) with $p, q \in \mathbf{Q}$. The positivity predicate is defined by $\mathsf{Pos}((p, q)) \equiv p < q$, and the cover $\lhd$ is defined inductively by the following rules (which are a formulation in our context of Joyal axioms, cf. [21, pp. 123–124]):

$$\frac{q \leqslant p}{(p,q) \lhd U}, \qquad \frac{(p,q) \in U}{(p,q) \lhd U},$$

$$\frac{p' \leqslant p < q \leqslant q' \qquad (p',q') \lhd U}{(p,q) \lhd U},$$

$$\frac{p \leqslant r < s \leqslant q \qquad (p,s) \lhd U \quad (r,q) \lhd U}{(p,q) \lhd U},$$

$$\infty \qquad \frac{wc((p,q)) \lhd U}{(p,q) \lhd U},$$

where in the last axiom we have used the abbreviation $wc((p,q)) \equiv \{(p',q') : p < p', q' < q\}$. (where *wc* stands for 'well-covered'). This presentation of the cover is essentially due to Coquand. The *formal reals* are just the formal points of such a formal topology.

We have then the following normal form theorem, by which the 'infinitary' rule ∞ is isolated:

Theorem of canonical form. Any derivation of a statement $a \lhd U$ can be brought to a form where the only application of the rule ∞ is the last one, just above the conclusion.

In this way the finitary part of the cover is distinguished from its infinitary component, and the logical tool we make use of is limited to a *finitary* inductive definition. The proof is by induction on the derivation of $a \lhd U$, as standard in proof theory. If $\lhd_\omega$ is the (finitary) compactification of $\lhd$, which by the remarks in the previous section coincides with the cover generated by the rules above except ∞, this amounts to have proved that

$$(p,q) \lhd U \text{ if and only if } wc((p,q)) \lhd_\omega U,$$

providing thus a definition of $(p,q) \lhd U$ as $wc((p,q)) \lhd_\omega U$, that is an elementary definition over a finitary inductive definition.

I express here the expectation that a similar (proof-theoretic) procedure can be used to separate the infinitary content of a cover from its finite part for a wider class of topologies (which presumably should be compact in some sense; cf. for instance [6]). This is still an open problem.

The above notion of well-covered elements can be generalized to an arbitrary formal topology, by setting

$$wc(a) \equiv \{b : S \lhd b^* \cup \{a\}\},$$

where $b^* \equiv \{c : c \downarrow b \lhd \emptyset\}$ is the subset of neighbourhoods which are apart from b. Then $b \,\varepsilon\, wc(a)$ is classically equivalent to saying that $\mathsf{ext}(b)$ is well covered by $\mathsf{ext}(a)$ if the closure of $\mathsf{ext}(b)$ is contained in $\mathsf{ext}(a)$. This brings us to define regular formal topologies as those topologies in which $a \lhd wc(a)$ for any a. It can be shown that such definition has some of the properties one would expect. For instance, one can prove that for any two formal points α and β, if $\alpha \subseteq \beta$ then $\alpha = \beta$, that is, the ordering on formal points is discrete (a paper with Curi is in preparation; see also [13]).

1.3.7. Classical theorems constructivized

A natural and reasonable question is of course how many of the classical theorems of (classical) topology can be obtained in the framework of formal topology. I am firmly convinced that, as with any form of constructive mathematics, the fact that relatively few results have been found up to now is not due to intrinsic obstacles, but mainly to the relatively little research energy which has been put in finding them.

Two important examples making explicit use of results of formal topology are Tychonoff's theorem [29] (building on previous work in [8]) and the Hahn-Banach theorem [7]. In [14], by introducing elementary diameters, a predicative version of Urysohn's metrization theorem is obtained.

More generally, formal topology is one of the ingredients of a new phase in the constructivization of classical mathematics, which is visible in the recent work by Coquand and others. Since no summary would do justice to this, I encourage the readers to look directly at his papers.

2. Some points, some novelties: the approach via the basic picture

Though successful, the definition of formal topology, as given in [32] or in Section 1.2, still leaves something to be desired. One desire is a convincing definition of formal closed subsets. Another is to avoid the operation $\cdot$ of formal intersection, which makes the treatment of some important examples, like $\mathscr{P}X$ and upper subsets in a preorder, a bit artificial. A positive solution to both requests (see [34]) has come from a deeper analysis of the notion of topological space. This has actually brought up much more than that. In fact, a whole new ground structure has emerged, which I have called the basic picture, since it shows how the main definitions of topology are deeply rooted to very basic ingredients, such as symmetry and logical duality. Topology, either with or without points, turns out to be obtainable simply by adding a principle of additivity of approximations (expressed by B2, in Section 2.1, and by $\downarrow$-Right, in Section 2.4), that is adding a notion of convergence. This in my opinion gives a very satisfactory explanation of the ground concepts of topology, which is independent of any foundational theory.

The basic picture has, moreover, a precise mathematical *raison d'être*, which has recently started to become clear. In fact, it seems that it is characterizable as the theory of what remains invariant under transfer along a continuous relation, in the same sense as topology can be seen as the theory of invariance under transfer along a continuous function.

I give here a short introduction to the ideas and to the main definitions of the basic picture. Some of them first appeared in [38]. For a complete development, see the series of papers in preparation [35,37,17–19], and other to follow. For some general observations connected with the basic picture, see Section 3, in particular Sections 3.1.4 and 3.2.

2.1. From concrete spaces to basic pairs

Let us resume our analysis of the notion of topological space, in Section 1.1, and more precisely at the moment in which we assumed the base to be closed under intersection. We now see that this is not necessary, and that actually relaxing that assumption allows one to see a simpler and deeper structure.

So assume, as before, that X is a set (of points), S is a set of indexes, and ext is a function from S into subsets of X. We consider all the subsets obtained by union, that is, all subsets of X of the form $\mathsf{ext}(U) \equiv \bigcup_{a \,\varepsilon\, U} \mathsf{ext}(a)$ for some $U \subseteq S$. Then we want to find out under which conditions on ext the subsets $\mathsf{ext}(U), U \subseteq S$, form a topology, that is, satisfy $\mathcal{O}1 - \mathcal{O}3$.

To this end, it is convenient to adopt a notation better suited than ext, as we now explain. Since a subset of X is nothing but a propositional function over X, a family of subsets $\mathsf{ext}(a) \subseteq X$ $(a \in S)$ is nothing but a propositional function with two arguments, one in X and one in S, in other words a binary relation between X and S (see Section 1.3.1). Then it is better to write such a relation as

$$x \Vdash a \ prop \ (x \in X, a \in S)$$

and to define ext in terms of it, by setting

$$\mathsf{ext}(a) \equiv \{x \in X\colon \ x \Vdash a\} \quad \text{for any } a \in S.$$

In this way the abstraction is kept at a lower level, both intuitively and formally (since $\mathsf{ext}\, a$ is obtained from $x \Vdash a$ by abstraction on x). Elements of S are called formal basic neighbourhoods, or more briefly observables, and $x \Vdash a$ is read as "x lies in a", or "x satisfies a", or more neutrally "x forces a". The choice of the name ext should then be clear: $\mathsf{ext}(a)$ is called the *extension* of the observable a. The notation with $\Vdash$ is extended to subsets by setting

$$x \Vdash U \equiv (\exists b \in S)(x \Vdash b \,\&\, b \,\varepsilon\, U) \equiv x \,\varepsilon \bigcup_{b \,\varepsilon\, U} \mathsf{ext}(b)$$

which agrees with the reading "x lies in U" since $\mathsf{ext}(U) \equiv \bigcup_{b \,\varepsilon\, U} \mathsf{ext}(b) \equiv \{x : x \Vdash U\}$. It is easy to check, at any desired level of formal details (using the definitions of [39] repeated in Section 1.3.1), that the family of subsets $\mathsf{ext}(U) \subseteq S$ $(U \subseteq S)$ is closed under unions. By this we mean, of course, that for any family of subsets $U_i \subseteq S$ $(i \in I)$ indexed by a set I, it holds that $\bigcup_{i \in I} \mathsf{ext}(U_i) = \mathsf{ext}(\bigcup_{i \in I} U_i)$. In fact, $x \,\varepsilon \bigcup_{i \in I} \mathsf{ext}(U_i) \equiv (\exists i \in I)(\exists b \,\varepsilon\, U_i)(x \Vdash b)$ is equivalent to $(\exists b \,\varepsilon \bigcup_{i \in I} U_i)(x \Vdash b) \equiv x \,\varepsilon\, \mathsf{ext}(\bigcup_{i \in I} U_i)$. So $\mathcal{O}3$ is automatically satisfied.

Condition $\mathcal{O}1$ also is easily expressed. In fact, $\emptyset = \mathsf{ext}(\emptyset)$ because $a \,\varepsilon\, \emptyset$ holds for no a, and $X = \mathsf{ext}(U)$ for some $U \subseteq S$ is equivalent to $X = \mathsf{ext}(S)$, that is

B1 $\quad x \Vdash S$ for any $x \in X$.

We thus can concentrate on $\mathcal{O}2$. If we express it without care, writing

$$(\forall U, V \subseteq S)(\exists W \subseteq S)(\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, W),$$

again an impredicative quantification comes up. However, this luckily is not really necessary. The quantification of the form $\forall U, V \exists W$ is solved if we find a uniform method which associates a subset W satisfying $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, W$ with any pair of subsets U, V. The simplest such method is to pick the largest among the open subsets contained in $\mathsf{ext}\, U \cap \mathsf{ext}\, V$. That is, if $\mathsf{ext}\, U \cap \mathsf{ext}\, V$ is open, which means that it is equal to $\mathsf{ext}\, W$ for some W, then it is bound to be equal to $\mathsf{ext} Z$ where Z is formed by all $c \in S$ whose extension is contained in $\mathsf{ext}\, U \cap \mathsf{ext}\, V$, in symbols $Z \equiv \{c \in S\colon\ \mathsf{ext}(c) \subseteq \mathsf{ext}\, U \cap \mathsf{ext}\, V\}$. So $\mathcal{O}2$ is equivalent to $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext} Z$. However, we can do much better than this. If we apply the same idea to open subsets of the form $\mathsf{ext}\, a$ with $a \in S$, we obtain

$$\text{B2} \qquad \mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}\, (a \downarrow b),$$

where $a \downarrow b \equiv \{c \in S\colon\ \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}$ is the largest subset whose extension is contained in $\mathsf{ext}\, a \cap \mathsf{ext}\, b$. It is now easy to see, by the distributivity property of $\mathcal{P}X$, that B2 is the right condition. In fact for any $U, V \subseteq S$ we have

$$\begin{aligned}
\mathsf{ext}\, U \cap \mathsf{ext}\, V &\equiv \left(\bigcup_{a\,\varepsilon\, U} \mathsf{ext}\, a\right) \cap \left(\bigcup_{b\,\varepsilon\, V} \mathsf{ext}\, b\right) \quad \text{by definition of } \mathsf{ext} \text{ on subsets,}\\
&= \bigcup_{a\,\varepsilon\, U} \bigcup_{b\,\varepsilon\, V} (\mathsf{ext}\, a \cap \mathsf{ext}\, b) \quad \text{by distributivity of } \mathcal{P}X,\\
&= \bigcup_{a\,\varepsilon\, U} \bigcup_{b\,\varepsilon\, V} \mathsf{ext}\, (a \downarrow b) \quad \text{by B2},\\
&= \mathsf{ext}\left(\bigcup_{a\,\varepsilon\, U} \bigcup_{b\,\varepsilon\, V} a \downarrow b\right) \quad \text{because } \mathsf{ext} \text{ distributes over unions.}
\end{aligned}$$

So we put

$$U \downarrow V \equiv \bigcup_{a\,\varepsilon\, U} \bigcup_{b\,\varepsilon\, V} a \downarrow b.$$

If B2 holds, then also $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, (U \downarrow V)$ holds, and hence $\mathcal{O}2$ is satisfied. Note that now $U \downarrow V$ is not necessarily the largest subset Z as defined above. But this is irrelevant. In fact, if $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, (U \downarrow V)$, then also $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, Z$ because $\mathsf{ext}\, (U \downarrow V) \subseteq \mathsf{ext}\, Z \subseteq \mathsf{ext}\, U \cap \mathsf{ext}\, V$.

The reason for names B1 and B2 is that they are just a compound expression, in our language, of the standard conditions for bases for a topology (see e.g. [15, p. 38]). B1 is clear: it says that the whole X is open. The inclusion $\mathsf{ext}\, (a \downarrow b) \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b$ of B2 always holds, and the other can be written as

$$\forall x (x \Vdash a \,\&\, x \Vdash b \rightarrow (\exists c)(x \Vdash c \,\&\, \mathsf{ext}\, (c) \subseteq \mathsf{ext}\, (a) \cap \mathsf{ext}\, (b)),$$

that is, for any point x lying in the basic neighbourhoods $\mathsf{ext}\, (a)$ and $\mathsf{ext}\, (b)$, there is a neighbourhood $\mathsf{ext}\, (c)$ of x which is contained both in $\mathsf{ext}\, (a)$ and in $\mathsf{ext}\, (b)$.

So we have proved that the collection of subsets $\mathsf{ext}\, U \subseteq X$ $(U \subseteq S)$, where $\mathsf{ext}\, U \equiv \bigcup_{a\,\varepsilon\, U} \mathsf{ext}(a)$, is a topology on X, that is, it satisfies $\mathcal{O}1 - \mathcal{O}3$, if and only if ext is a base, that is, it satisfies B1 and B2 above.

We have thus reached a definition of concrete space (see Section 1.1) which is free of the operation $\cdot$ of formal intersection, as we wished. To help the intuition, we express B1 and B2 in the notation with $\Vdash$.

Definition 3. A concrete space is a structure $\mathcal{X} = (X, \Vdash, S)$ where

X is a set, whose elements $x, y, z, \ldots$ are called concrete points;

S is a set, whose elements $a, b, c, \ldots$ are called observables, or formal basic neighbourhoods;

$\Vdash$ is a binary relation from X to S, called forcing, which satisfies

B1 $\quad x \Vdash S \quad$ for any $x \in X$,

B2 $\quad \dfrac{x \Vdash a \qquad x \Vdash b}{x \Vdash a \downarrow b} \quad$ for any $a, b \in S$ and $x \in X$.

This brings us easily to a new formulation of the notion of formal topology, which is obtained from Definition 1 by suppressing $\cdot$, 1 and their axioms, and by replacing them with the single condition (which expresses B2 in formal terms)

$$\downarrow\text{-Right} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V},$$

where $U \downarrow V \equiv \{d\colon\ (\exists b\,\varepsilon\, U)(d \lhd \{b\})\,\&\,(\exists c\,\varepsilon\, V)(d \lhd \{c\})\}$. This variant on the definition (see also [42] for a similar approach) has been adopted in [10] since it allows a smoother approach to the topic of inductive generation. Note also that now both $\mathcal{P}S$ and the collection of upper subsets of a preordered set $(S, \leqslant)$ fall easily and naturally under the definition of formal topology. Moreover, it can be proved that for any formal topology $\mathcal{S}$ with $\downarrow$ there is a formal topology $\mathcal{S}'$ with $\cdot$ (as in Section 1.2) such that $\mathcal{S}$ and $\mathcal{S}'$ produce the same frame of formal open subsets. The condition $\downarrow$-Right is present also in the new definition of formal topology which will be given in Section 2.4.

These are useful technical improvements. However, the most important consequence of the analysis which led to Definition 3 above is conceptual, rather than technical. At an impredicative reading, the above definition of concrete space is just a cumbersome formulation, but perfectly equivalent to the usual definition of topological space. Predicatively, the notion of set is much stricter, and hence many examples of spaces do not fall under Definition 3 simply because the collection of points X is not a set: this is a good reason to develop formal topology. Nevertheless, although keeping this crucial remark in mind, one can see that the framework provided by Definition 3 is sufficient to define the notions of open and closed subset in a way which is perfectly acceptable also constructively. In fact, as we will see, the way to dispense with the powerset axiom and second-order quantifications is to reduce systematically to quantifications over basic neighbourhoods, that is over the set S. Thus the set S is an essential ingredient of the definition, and it should not be forgotten (in the sense of Section 3.1.3), contrary to the common approach which tends to avoid any reference to bases.

The usual definition can be rephrased by saying that a subset $E \subseteq X$ is *open* if: whenever $x \,\varepsilon\, E$, then this is true in a continuous way, that is not only x, but also a whole neighbourhood of x is contained in E. In our notation this becomes

$$x \,\varepsilon\, E \to (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\ a \subseteq E).$$

We put as usual $\mathsf{int}\ E \equiv \{x \in X\colon\ (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\ a \subseteq E)\}$. Such operator int, for interior, can be thought of intuitively as a *rejector*, or *thinner*, which makes E as thin as possible, that is, which throws away from E all isolated points, but is unable to throw away from E a whole neighbourhood $\mathsf{ext}\ a$. So E is open if $E \subseteq \mathsf{int}\ E$, which is equivalent to saying that the rejector operator has no effect on E.

The definition of closed subset can be put in perfectly dual terms. In fact, the usual definition can be expressed by saying that $D \subseteq X$ is closed if whenever it is continuously satisfiable for x to be in D, then actually $x \,\varepsilon\, D$. I here say that $x \,\varepsilon\, D$ is continuously satisfiable if any neighbourhood of x touches D. We now can see that the notion of meet $≬$ begins to be useful. In fact, the above intuitive definition is formally expressed by

$$(\forall a \in S)(x \Vdash a \to \mathsf{ext}\ a \between D) \to x \,\varepsilon\, D.$$

The subset $\mathsf{cl}\, D \equiv \{x\colon\ (\forall a \in S)(x \Vdash a \to \mathsf{ext}\ a \between D)\}$ is the *closure* of D, and one can intuitively think of cl as an *attractor*, or *fattener* operator: it adds to D all points x which "continuously touch" D, in the sense that any neighbourhood of x meets D. Note that this is a positive way of affirming that x cannot be continuously separated from D, which would be $\neg\exists a(x \Vdash a \,\&\, \mathsf{ext}(a) \cap D = \emptyset)$ and which is equivalent to $\forall a(x \Vdash a \to \mathsf{ext}(a) \between D)$ only classically. So D is closed if the attractor operator cl has no effect on D, that is, D is already as big as it is consistent to be.

The notation we adopted, together with explicit expression of the logical formalism involved, allows one to see immediately the strong logical relation between interior and closure. The definition of closure is logically dual to that of interior, in the sense that $\exists$ is replaced by $\forall$, & is replaced by $\to$ (which in type theory are special cases of $\exists$ and $\forall$, respectively) and $\subseteq$ is replaced by $\between$ (whose definitions are in turn obtained one from the other by interchanging $\forall$ with $\exists$). We want to keep this duality, and actually build on it and make it clearer. Adopting classical logic here would immediately reduce it to the much simpler duality between a subset and its complement. In fact, by classical logic we would have: D *closed* $\equiv (\forall a \in S)(x \Vdash a \to \mathsf{ext}\ a \between D) \to x \,\varepsilon\, D$ if and only if $\neg(\exists a \in S)(x \Vdash a \,\&\, \neg(\mathsf{ext}\ a \between D)) \to x \,\varepsilon\, D$ if and only if $x \varepsilon {-D} \to (\exists a \in S)(x \Vdash a \,\&\, \mathsf{ext}\ a \subseteq -D) \equiv -D$ *open*.

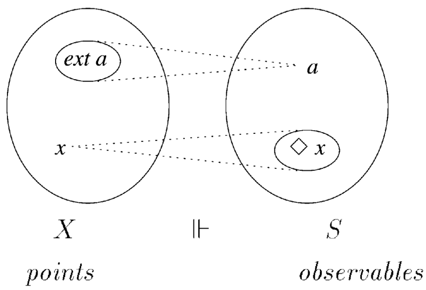
So, in the same way as classical logic reduces the meaning of existential quantification to a negation of a universal quantification, here it would reduce the definition of closed subset, which in the essence is a quantification of the form $\forall\exists$, to that of open subset, which is of the form $\exists\forall$.

An obvious remark, which however is of crucial importance for what follows, is that the conditions B1 and B2 have no role in the definitions of open and of closed subsets. Then it is worthwile to analyse the logical duality between closure and interior in the more general structure given simply by two sets X, S and any binary relation

$\Vdash$ between them. I call it a *basic pair*. Moreover, the simple remark that the notion of basic pair is perfectly self-symmetric, will lead to the discovery of the role of symmetry in topology.

2.2. A structure for topology

From now on, we keep the sets X and S always fixed, also in the sense that we think of X as situated at the left, and of S as situated at the right as in Picture 1 below.



Picture 1

We systematically use $x, y, \ldots$ for elements and $D, E, \ldots$ for subsets of X, $a, b, \ldots$ for elements and $U, V, \ldots$ for subsets of S. In this way we can avoid to mention the domain of quantifications, and we shall do so from now on. One can think intuitively of $x, y, \ldots$ as points and of $a, b, \ldots$ as observables (cf. [43]), so that $x \Vdash a$ means that the observable a applies to the point x.

The relation $x \Vdash a$ is expressed at the left by the synonym $x \,\varepsilon\, \mathsf{ext}\ a$, where $\mathsf{ext}\ a \equiv \{x :\ x \Vdash a\}$ is the extension of a, and at the right by the synonym $a \,\varepsilon \diamond x$, where $\diamond x \equiv \{a : x \Vdash a\}$. The relation $\Vdash$ induces four monotone operators on subsets; in the language of categories, these are just functors from $\mathscr{P}X$ into $\mathscr{P}S$ or conversely, when both $\mathscr{P}X$ and $\mathscr{P}S$ are seen as preordered categories. First we define the functors ext and rest from $\mathscr{P}S$ into $\mathscr{P}X$ by setting:

$$x \varepsilon\ \mathsf{ext}(U) \equiv \diamond x \between U,$$
$$x \varepsilon\ \mathsf{rest}(U) \equiv \diamond x \subseteq U.$$

These are, respectively, just the definitions of weak, or existential, and of strong, or universal, anti-image of the subset U along the relation $\Vdash$. The name rest is due to the idea of conceiving $\mathsf{rest}\ U$ as the *restriction* to those points of X which live in U, in the sense that all their observables belong to U. If the relation from X to S is denoted more simply by R, or even better by a small r (because we will think of it also as a function from X into $\mathscr{P}S$, and not only as a binary propositional function), a good notation is r^- for the weak, and r^* for the strong anti-image. That is, using the notation $rx \equiv \{a : xra\}$ for the r-image of x (which is $\diamond x$ when r is denoted by $\Vdash$), we put

$$x \,\varepsilon\, r^-(U) \equiv rx \between U,$$
$$x \,\varepsilon\, r^*(U) \equiv rx \subseteq U.$$

An important little observation, which will often be used tacitly, is that the existential anti-image is just the union of anti-images of elements, that is $\mathsf{ext}(U) \equiv \cup_{b \varepsilon U}\, \mathsf{ext}\ b$; note also that this gives in particular $\mathsf{ext}(\{b\}) = \mathsf{ext}\ b$, and this is why we can use the same letter ext without confusion both for the operator on elements and for that on subsets. Note also that r^- and r^* coincide when r is the graph of a function, because $rx \between U$ if and only if $rx \subseteq U$ when rx is a singleton.

The same definitions apply also to the inverse relations. So we have two functors $\Diamond$ and $\Box$ from $\mathscr{P}X$ into $\mathscr{P}S$ which are defined by [10]

$$a \,\varepsilon\, \Diamond D \equiv \mathsf{ext}\ a \between D,$$
$$a \,\varepsilon\, \Box D \equiv \mathsf{ext}\ a \subseteq D.$$

Note that, as for ext, $\Diamond(\{b\}) = \Diamond b$. In the abstract notation with r, we write r^- for the relation which is inverse of r, that is, which is defined by $ar^-x \equiv xra$ and also extend to r^- the notation for images of elements, so that $r^-a \equiv \{x : xra\}$; this notation is justified since the r^--image of the element a coincides with the weak anti-image of the singleton subset $\{a\}$ along r as defined before, that is $r^-a = r^-\{a\}$. Then we can put:

$$a \,\varepsilon\, rD \equiv r^-a \between D,$$
$$a \,\varepsilon\, r^{-*}D \equiv r^-a \subseteq D.$$

Note that the weak anti-image of U along r, as defined before, coincides with the (direct) image of U along the inverse relation r^-, and so both are denoted by r^-U. As before for ext and r^-, one can see also that $r\{x\} = rx$ and that $rD = \cup_{x \varepsilon D}\, rx$.

The starting point of the basic picture is the discovery that the operators int and cl as defined in the preceding section are nothing but the composition of the operators just defined:

$$\mathsf{int} \equiv \mathsf{ext}\ \Box \qquad \text{and} \qquad \mathsf{cl} \equiv \mathsf{rest}\ \Diamond.$$

In fact, one can easily see that $x \,\varepsilon\, \mathsf{int}\ D \equiv \exists a(x \Vdash a \,\&\, \mathsf{ext}\ a \subseteq D) \equiv \Diamond x \between \Box D$ and that $x \,\varepsilon\, \mathsf{cl}\ D \equiv \forall a(x \Vdash a \rightarrow \mathsf{ext}\ a \between D) \equiv \Diamond x \subseteq \Diamond D$. So one can see that the duality between int and cl is the result of a deeper duality between $\Diamond$ and $\Box$, and between ext and rest.

This is a good point to repeat that the structure consisting of X, $\Vdash$ and S is absolutely symmetric. Maybe it takes some effort to abandon the intuition of X as points and of S as observables, but the plain mathematical content is only that they are two sets linked

[10] Clearly, the signs $\Diamond$ and $\Box$ are taken from modal logic; if $S = X$ and $\Vdash$ is the accessibility relation, then $\Diamond D$ and $\Box D$ are the valuations of formulae $\Diamond\phi$ and $\Box\phi$, respectively, if D is the valuation of ϕ. The operators ext and rest then correspond to possibility and necessity in the past, respectively, as in temporal logic.

by an arbitrary relation. So, in a fully symmetric way we can define two operators on $\mathcal{P}S$, which are symmetric of int and of cl, respectively:

$$\mathcal{J} \equiv \Diamond\ \mathsf{rest} \qquad \text{and} \qquad \mathcal{A} \equiv \Box\ \mathsf{ext}.$$

In fact, they are obtained by replacing ext, $\Box$, rest, $\Diamond$ with their symmetric $\Diamond$, rest, $\Box$, ext, respectively. The meaning of such operators [11] becomes clearer by making definitions explicit. Since $a\,\varepsilon\,\mathcal{A}U \equiv \mathsf{ext}\ a \subseteq \mathsf{ext}\ U \equiv \forall x(x \Vdash a \to x \Vdash U)$, then $a\,\varepsilon\,\mathcal{A}U$ means that all points lying in a also lie in U. So, $a\,\varepsilon\,\mathcal{A}U$ is something we know already, since it expresses the intuition of the formal cover $a \lhd U$, as in Section 1.1.

Let us turn to $a\,\varepsilon\,\mathcal{J}U \equiv \mathsf{ext}\ a \between \mathsf{rest}(U)$. The explicit definition is $\exists x(x \Vdash a \,\&\, \Diamond x \subseteq U)$, which means that a is inhabited by some point, about which we know in addition that all its neighbourhoods are in U. Informally, $a\,\varepsilon\,\mathcal{J}U$ says that there is a point in $\mathsf{ext}\ a$, and U gives positive information on where inside $\mathsf{ext}\ a$ it is. In the special case $U = S$, $a\,\varepsilon\,\mathcal{J}S$ means simply that $\mathsf{ext}\ a$ is inhabited; we met this in Section 1.1 as the intuitive explanation of the predicate Pos. So $a\,\varepsilon\,\mathcal{J}U$ is the pointwise definition of a new formal relation between an element a and a subset U of S; we denote it by $\mathsf{Pos}(a,U)$, or also by $a \ltimes U$, and call it a binary positivity predicate. As it is evident from the preceding explanation, the idea of introducing $\mathcal{J}$ or $\ltimes$ is quite natural by *structural* reasons: symmetry, since $\mathcal{J}$ is symmetric to int, and logical duality, since $\mathcal{J}$ is dual to $\mathcal{A}$. Whatever is the way to reach it, however, it gives a new possible choice of primitive relation on S, namely $\ltimes$, to be added to $\lhd$. So following the method in Section 1.1 one is lead to a new definition of formal topology, with a binary positivity predicate, see Section 2.4. This is one of the main conceptual novelties of the present approach. Also, since $\mathcal{A}$ and $\mathcal{J}$ can be defined on *any* basic pair, one can apply the same method on an arbitrary basic pair and obtain a weaker notion than that of formal topology, see Section 2.4. This is another important conceptual novelty. Some comments will be given after the mathematical development.

Since the operators are defined in terms of a relation, through existential and universal quantifications, it follows that there is an adjunction between each existential operator and the universal operator in the opposite direction. So ext is left adjoint of $\Box$ and $\Diamond$ is left adjoint of rest:

$$\begin{array}{ll} \mathsf{ext} \dashv \Box & \text{that is } \mathsf{ext}\ U \subseteq D \text{ if and only if } U \subseteq \Box D, \text{ for any } D, U, \\ \Diamond \dashv \mathsf{rest} & \text{that is } \Diamond D \subseteq U \text{ if and only if } D \subseteq \mathsf{rest}\ U, \text{ for any } D, U. \end{array}$$

A formal proof is based on the equivalence between $\exists x Ax \to B$ and $\forall x(Ax \to B)$, in intuitionistic logic. In the notation with r, these are just the adjunctions:

$$\begin{array}{ll} r^- \dashv r^{-*} & \text{that is } r^- U \subseteq D \text{ if and only if } U \subseteq r^{-*}D, \text{ for any } D, U, \\ r \dashv r^* & \text{that is } rD \subseteq U \text{ if and only if } D \subseteq r^* U, \text{ for any } D, U, \end{array}$$

[11] The choice of the letter $\mathcal{J}$ is due to the fact that I had no other available, and it should not be connected with the so called j-operators of locale theory, see [21].

respectively. I call these the two *fundamental adjunctions* determined by the relation r.

It is a general well known fact that the composition of the left adjoint (existential) after the right adjoint (universal) operator gives an interior operator. So $\mathscr{J} \equiv \Diamond\mathsf{rest}$ is an interior operator; this means that $\mathscr{J}$ satisfies $\mathscr{J}U \subseteq U$, $U \subseteq V \rightarrow \mathscr{J}U \subseteq \mathscr{J}V$ and $\mathscr{J}U \subseteq \mathscr{J}\mathscr{J}U$, or equivalently $\mathscr{J}U \subseteq U$ and $\mathscr{J}U \subseteq V \rightarrow \mathscr{J}U \subseteq \mathscr{J}V$. By symmetry, $\mathsf{int} \equiv \mathsf{ext}\,\Box$ also is an interior operator. Note that *i*. int is proved to be an interior operator on any basic pair (thus also when B1 and B2 are *not* assumed) and hence *ii*. int does not in general preserve finite intersections (one can prove that this is actually equivalent to B2), that is, it is *not* what is sometimes called a *topological* interior operator (see e.g. [43]).

Similarly, the composition of $\Box$ after ext, namely $\mathscr{A}$, and of rest after $\Diamond$, namely cl, are closure operators. This means that $U \subseteq \mathscr{A}U$, $U \subseteq V \rightarrow \mathscr{A}U \subseteq \mathscr{A}V$ and $\mathscr{A}\mathscr{A}U \subseteq \mathscr{A}U$ hold, or equivalently $U \subseteq \mathscr{A}U$ and $U \subseteq \mathscr{A}V \rightarrow \mathscr{A}U \subseteq \mathscr{A}V$. Similarly for cl; of course, two remarks analogous to those on int apply to cl.

For a closure operator, such as $\mathscr{A}$, we say that a subset $U \subseteq S$ is $\mathscr{A}$-saturated if $U = \mathscr{A}U$. So $D \subseteq X$ is cl-saturated if $D = \mathsf{cl}\ D$, that is when D is closed. We denote by *Sat*($\mathscr{A}$), and *Sat*(cl), the collection of saturated subsets.

Similarly, for an interior operator, such as $\mathscr{J}$, we say that a subset $U \subseteq S$ is $\mathscr{J}$-reduced if $U = \mathscr{J}U$. So $D \subseteq X$ is int-reduced if $D = \mathsf{int}\ D$, that is when D is open. The collections of reduced subsets are denoted by *Red*($\mathscr{J}$) and *Red*(int).

For any operator $\mathscr{C}$, either a closure or an interior operator, one can define suprema and infima by putting

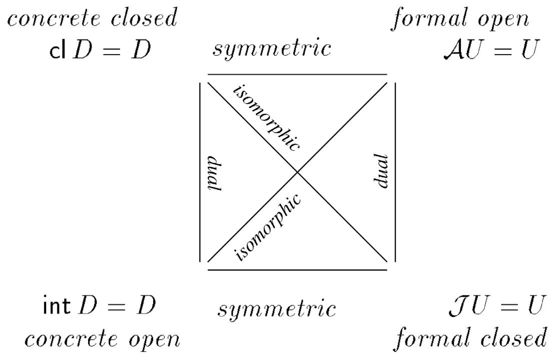
$$\bigvee_{i\in I} \mathscr{C}U_i \equiv \mathscr{C}\left(\bigcup_{i\in I} \mathscr{C}U_i\right) \quad \text{and} \quad \bigwedge_{i\in I} \mathscr{C}U_i \equiv \mathscr{C}\left(\bigcap_{i\in I} \mathscr{C}U_i\right).$$

So *Sat*($\mathscr{A}$), *Sat*(cl), *Red*($\mathscr{J}$) and *Red*(int) are all complete lattices. It is not difficult to prove (by making systematic use of the two fundamental adjunctions) that actually *Red*(int) is isomorphic to *Sat*($\mathscr{A}$), via the isomorphism $\Box$: *Red*(int) $\rightarrow$ *Sat*($\mathscr{A}$) with inverse ext: *Red*(int) $\leftarrow$ *Sat*($\mathscr{A}$). This is why $\mathscr{A}$-saturated subsets are called *formal open*, and int-reduced subsets, viz. open subsets of X, are called concrete open when there is danger of confusion.

The isomorphism between formal open and concrete open subsets was somehow expected, see the ideas in Section 1.1. What came as a surprise is the fact that to be able to obtain a similar isomorphism for concrete closed subsets one has to introduce a new primitive, namely $\mathscr{J}$ or $\ltimes$, and define a subset of S to be *formal closed* if it is $\mathscr{J}$-reduced.

Picture 2 sums up the situation. Note that in the top line we have two closure operators, which are of the form $\forall\exists$, while in the bottom line we have two interior operators, of the form $\exists\forall$. The choice of names is due to the fact that we want the two lattices of (concrete and formal) open subsets, and equally for closed subsets, to be isomorphic. This has the consequence that formal open subsets are described by a closure operator and formal closed subsets by an interior operator.

This concludes the first chapter of the basic picture (a full exposition is in [35]). We are now going to see that similar structural characterizations can be obtained also for other notions of topology.



Picture 2

2.3. The essence of continuity

A common definition says that a function $f: X \to Y$ is continuous if, for any $x \in X$, whatever neighbourhood E of fx one considers, there is a whole neighbourhood D of x which is all sent "close" to fx, that is inside E. In our framework, assume X and Y are the sets of points of two basic pairs or concrete spaces $X \overset{\Vdash_1}{\to} S$ and $Y \overset{\Vdash_2}{\to} T$ (we will omit subscripts unless strictly necessary). Then the definition of continuity for f is formally expressed by

$$\forall b(f\,x \Vdash b \to \exists a(x \Vdash a \ \&\ \forall z(z \Vdash a \to f\,z \Vdash b)). \tag{1}$$

As it is well known, f is continuous if and only if the inverse-images along f of open subsets of Y are open in X. In our framework, this means that for each $b \in T$, $f^{-}\mathsf{ext}\ b = \mathsf{ext}(\{a \in S: \mathsf{ext}\ a \subseteq f^{-}\ \mathsf{ext}\ b\})$. If we define a relation $s: S \to T$ by putting $a\,s\,b \equiv \mathsf{ext}\ a \subseteq f^{-}\ \mathsf{ext}\ b$, this equation means that $f^{-} \circ {\Vdash_2}^{-} = {\Vdash_1}^{-} \circ s^{-}$. But then, to restore symmetry, one is lead to generalize the treatment to a *relation* r also from X to Y. This move is of crucial importance, since it allows to make the structure underlying continuity more clearly visible, and simpler, than with functions.

Let us first find a suitable extension of (1) to relations. First, rewrite (1) as $\forall b(f\,x\,\varepsilon\ \mathsf{ext}\,b \to \exists a\ (x\,\varepsilon\ \mathsf{ext}\ a \,\&\, \mathsf{ext}\ a \subseteq f^{-}\ \mathsf{ext}\ b))$. At this point, recall that fx is an element, while rx is a subset of Y. So we can think of $f\,x\,\varepsilon\ \mathsf{ext}\ b$ as $\{fx\} \subseteq \mathsf{ext}\ b$ or as $\{fx\} \between \mathsf{ext}\ b$; the second choice works better. So we say that $r: X \to Y$ is continuous if

$$rx \between \mathsf{ext}\ b \to \exists a(x\,\varepsilon\,\mathsf{ext}\ a \& \mathsf{ext}\ a \subseteq r^{-}\ \mathsf{ext}\ b) \tag{2}$$

holds for any $x \in X$, $b \in T$.

An important discovery is the equivalence of the following conditions:
(a) r is continuous,
(b) r^{-} is open, that is $r^{-}\mathsf{ext}\ b$ is open in X for any $b \in T$,
(c) there exists s: $S \to T$ such that $r\,x \between \mathsf{ext}\ b \leftrightarrow \Diamond x \between s^{-}b$, for any $x \in X$, $b \in T$.

Note that the equivalence in c. is nothing but a way to express that $\Vdash \circ r = s \circ \Vdash$, that is commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Vdash} & S \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{\Vdash} & T \end{array}$$

So we define a morphism from $X \xrightarrow{\Vdash} S$ to $Y \xrightarrow{\Vdash} T$ to be a pair of relations $r : X \to Y$ and $s : S \to T$ which make the diagram commute. (r,s) is called a *relation-pair*. The presence of s in the definition has the purpose of keeping the information which otherwise is restored only by a quantification over relations, as in (c) above.

Commutativity of a diagram is the clearest structural description one can find. The framework of basic pairs shows that the essence of continuity is just a commutative square. In the case of functions, we obtain the usual definition as a special case.

Several other equivalent formulations of continuity are possible. Since commutativity of the diagram is equivalently expressed by $r^- \circ \Vdash^- = \Vdash^- \circ s^-$ and $r^* \circ \Vdash^* = \Vdash^* \circ s^*$, the notion of relation-pair is equivalently presented by each of the equations

$$r^- \ \mathsf{ext}\ V = \mathsf{ext}\ s^- V \text{ for any } V \subseteq T,$$
$$r^* \ \mathsf{rest}\ V = \mathsf{rest}\ s^* V \text{ for any } V \subseteq T.$$

The first says that r^- is open, and s^- is a method by which we determine the open subsets of X which are the existential anti-images of open subsets of Y along r. The second says that r^* is closed, and s^* gives a method by which we determine the closed subsets of X which are universal anti-images of closed subsets of Y along r. In other words, s gives the method by which we know that r^- is open and r^* is closed. Even if, given r, one can define a relation s such that (r,s) is a relation-pair by putting $asb \equiv \mathsf{ext}\ a \subseteq r^- \mathsf{ext}\ b$, to "forget" s thinking that it can always be restored is not safe. For instance, only keeping the information s, some of the common equivalent characterizations of continuity, like r^- is open if and only if r^* is closed, can be proved constructively. If s is lost, by knowing that r^* is closed there is no way, not even impredicatively, neither to restore s, nor to prove that r^- is open; in fact, two finite counterexamples in [18] show that the two conditions are no longer equivalent when the formal side is forgotten.

The category **BP** with basic pairs as objects and relation-pairs as arrows differs from the well-known category $\mathbf{Rel}^2$, of relations and commutative square diagrams, only in the fact that equality between relation-pairs is explicitly defined. Two relation-pairs (r,s) and (r',s') are declared to be equal if they behave in the same way with respect to open and to closed subsets, both concrete and formal. This too turns out to be equivalent to a fully structural condition, namely that their top-left bottom-right diagonals coincide,

$$\begin{tikzcd} X \arrow[r, "\Vdash_1"] \arrow[d, shift right, "r"'] \arrow[d, shift left, "r'"] & S \arrow[d, shift right, "s'"'] \arrow[d, shift left, "s"] \\ Y \arrow[r, "\Vdash_2"] & T \end{tikzcd}$$

that is $\Vdash_2 \circ r = \Vdash_2 \circ r' = s \circ \Vdash_1 = s' \circ \Vdash_1$. The category **BP** is also different from the category of boolean Chu spaces (see [31]), since morphisms of Chu spaces are functions (one in the reverse direction of the other).

Finally, the intrinsic symmetry of basic pairs and of relation-pairs is formally expressed by the fact that the functor $(\,)^-$, defined by $(X, \Vdash, S)^- \equiv (S, \Vdash^-, X)$ and by $(r,s)^- \equiv (s^-, r^-)$, is a self-duality of **BP**.

I refer the readers to [17] for a detailed exposition of the content of this section, with complete proofs.

2.4. Basic topologies, formal topologies, formal spaces

The methodology to obtain the definition of a formal notion is always the same, and it has been described in Section 1.1 (see also Section 3.2.2). The difference is that now for this task we can make use of the preceding analysis of the structure induced on a basic pair, and hence also on a concrete space. So on one hand it is easier, and on the other hand it produces a richer structure. First we introduce a new notion, namely that of (formal) basic topology, which is obtained by describing the structure induced on the formal side of a basic pair, and by taking the result as an axiomatic definition. The new definition of formal topology is then obtained simply by adding a formal condition expressing that the basic pair is actually a concrete space. Finally, the notion of formal point is obtained as an axiomatic description of the subset $\Diamond x$ determined by a concrete point x on the formal side.

There are a few good reasons to do all this, that is, to study formal topology: the first is that it is a natural way, and often the only one, to be able to deal predicatively with certain spaces. After all, this is just how the real numbers are obtained from the topology of rational intervals. The second reason is that it provides more general tools to topology (see Section 3.2.2). A third good reason to do it is simply that it can be done, and that nice new structures emerge in this way. Thus it contributes to expand the territory of mathematical thought (see Section 3.1.4).

In this section the definitions I propose will be introduced and shortly justified. The problem of the correctness of such definitions will be discussed in Section 2.6 in detail.

We have already seen that any basic pair $X \xrightarrow{\Vdash} S$ induces a closure operator $\mathscr{A} \equiv \Box\, \mathsf{ext}$ and an interior operator $\mathscr{J} \equiv \Diamond \mathsf{rest}$ on the formal side, namely on the set S. This is all we can say given that ext is left adjoint to $\Box$, and that $\Diamond$ is left adjoint to rest, respectively. What we have to add now is a condition linking $\mathscr{A}$ with $\mathscr{J}$ and expressing the fact that the two adjunctions $\mathsf{ext} \dashv \Box$ and $\Diamond \dashv \mathsf{rest}$ are induced by the *same* relation $\Vdash$. For any $a \in S$ and $U, V \subseteq S$, the rule

$$\frac{\mathsf{ext}\ a \between \mathsf{rest}\ V \quad \mathsf{ext}\ a \subseteq \mathsf{ext}\ U}{\mathsf{ext}\ U \between \mathsf{rest}\ V}$$

clearly holds. Since $\mathsf{ext}\ a \between \mathsf{rest}\ V \equiv a\ \varepsilon\ \mathcal{J}V$, $\mathsf{ext}\ a \subseteq \mathsf{ext}\ U \equiv a\ \varepsilon\ \mathcal{A}U$ and $\mathsf{ext}\ U \between \mathsf{rest}\ V$ iff $\exists b(b\ \varepsilon\ U\ \&\ b\ \varepsilon\ \mathcal{J}V)$, it says that $a\ \varepsilon\ \mathcal{A}U$ and $a \varepsilon \mathcal{J}V$ imply $U \between \mathcal{J}V$. Since the element a does not appear in the conclusion, the conclusion is valid simply if such an element exists. So we have the rule

$$\text{compatibility} \quad \frac{\mathcal{A}U \between \mathcal{J}V}{U \between \mathcal{J}V}.$$

Thus the first definition is simply that a *formal basic topology* is a triple $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ where S is a set, $\mathcal{A}$ is a closure operator, $\mathcal{J}$ is an interior operator, and they are linked by compatibility (note that compatibility is the same as the equivalence $\mathcal{A}U \between \mathcal{J}V \leftrightarrow U \between \mathcal{J}V$, since the direction $\leftarrow$ holds trivially). In the notation with $a \lhd U$ for $a\ \varepsilon\ \mathcal{A}U$ and $a \ltimes V$ for $a\ \varepsilon\ \mathcal{J}V$, this amounts to:

$$\text{reflexivity} \quad \frac{a\ \varepsilon\ V}{a \lhd V}, \quad \text{transitivity} \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V},$$

$$\text{co-reflexivity} \quad \frac{a \ltimes V}{a\ \varepsilon\ V}, \quad \text{co-transitivity} \quad \frac{a \ltimes U \qquad (\forall b)(b \ltimes U \rightarrow b\ \varepsilon\ V)}{a \ltimes V},$$

$$\text{compatibility} \quad \frac{a \ltimes V \quad a \lhd U}{U \ltimes V},$$

where we now add the shorthand $U \ltimes V$ for $(\exists b\ \varepsilon\ U)(b \ltimes V)$. It is just natural to carry over the terminology from basic pairs and say that U is formal closed if $U = \mathcal{J}U$ and formal open if $U = \mathcal{A}U$.

The intuitive meaning of compatibility is that any formal closed subset $V = \mathcal{J}V$ must split any cover, in the sense that if $a \lhd U$ and if $a\ \varepsilon\ V$, then V must proceed and meet U. This is nothing but the symmetric of the usual condition defining the concrete closure. To see this, first note that, if we apply the same methodology to the concrete side, since a basic pair is fully symmetric we obtain a fully symmetric definition: a *concrete basic topology* is a triple $(X, \mathsf{int}, \mathsf{cl})$ where X is a set, int is an interior operator, cl is a closure operator, and they are linked by

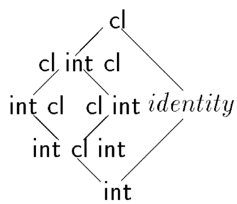
$$\mathsf{cl}\ D \between \mathsf{int}\ E \leftrightarrow D \between \mathsf{int}\ E.$$

This equivalence is nothing but the characterization of closure in terms of all open subsets, rather than subbasic neighbourhoods (as in Section 2.1).

The structure of a concrete basic topology is, from a purely mathematical point of view, just identical with that of a formal basic topology, and thus one could call each of them just a *basic topology*. But note that terminology is quite different, since a concrete open in $(X, \mathsf{int}, \mathsf{cl})$ is kept fixed by the interior operator int, while a formal open in $(S, \mathscr{A}, \mathscr{J})$ is kept fixed by the closure operator $\mathscr{A}$. This is what the adjective "concrete" or "formal" in front of "basic topology" recalls.

The definition of (concrete) basic topology is very simple, and should find its place together with other definitions weaker than that of topological space which were given long ago by Kuratowski, Frechet, Čech, and others. Its peculiarity is that it has a purely structural justification, and that it is meaningful only by assuming intuitionistic logic and a primitive notion of closed subset (see Section 2.6).

The fact that the definition of basic topology is not too weak is confirmed by some initial results on the structure of possible combinations of the operators int, cl and opposite $-$. First, one can easily prove that the different combinations of int and cl are exactly seven, and that the mutual inclusions are only [12] those shown in the picture (in which inclusion appears as an edge upwards):



Adding the equation $\mathsf{cl} = -\mathsf{int}-$ and classical logic, one can then easily obtain the well known result by Kuratowski telling that there are at most 14 different combinations of $-$, int and cl.

In the general case, from compatibility one can obtain that $\mathsf{cl}\ D \cap \mathsf{int}\ E = \emptyset \leftrightarrow D \cap \mathsf{int}\ E = \emptyset$, and using this one can derive that the equations linking $-$, int and cl are:

$$\mathsf{cl} - D \subseteq -\mathsf{int}\ D = \mathsf{cl} - \mathsf{int}\ D$$

$$\mathsf{int} - \mathsf{cl}\ D = \mathsf{int} - D \subseteq -\mathsf{cl} D$$

(the proofs were first given in [44]). It is easy to find out that the inclusions $-\mathsf{int}\ D \subseteq \mathsf{cl} - D$ and $-\mathsf{cl}\ D \subseteq \mathsf{int} - D$ do not hold in general. The above equations seem to express the basic properties of closure, interior and opposite in the intuitionistic case. However, it is still not known (to me) whether other inclusions or equations involving more occurrences of $-$, int and cl hold. An initial study in [44] has shown that all the different combinations with only one occurrence of $-$ do not exceed the number of 22, and I still don't know whether it is lower than that. With two occurrences of—, the number seems to get much higher. In general, it is apparently still an open problem even to decide whether the total number of combinations is finite or infinite.

It must be emphasized that all definitions and results so far have been obtained starting from an arbitrary basic pair. It is now a relatively easy matter to find a

[12] The method to find counterexamples for the other inclusions is interesting: one can choose a suitable basic pair, and use the logical expressions for int and cl to show that they would give some implications which are not valid intuitionistically.

formal condition corresponding to the property we called B2, that is $\mathsf{ext}\ U \cap \mathsf{ext}\ V = \mathsf{ext}(U \downarrow V)$. In fact, if we express it in the equivalent form $\forall x(x \Vdash U \,\&\, x \Vdash V \to x \Vdash U \downarrow V)$ we see that, by replacing an arbitrary concrete point x with an arbitrary observable a and the relation $\Vdash$ with the cover $\lhd$, we obtain $\forall a(a \lhd U \,\&\, a \lhd V \to a \lhd U \downarrow V)$. This is the same as the rule

$$\downarrow\text{-Right} \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}.$$

Note that the formal expression of $\mathsf{ext}\ b \cap \mathsf{ext}\ c = \mathsf{ext}(b \downarrow c)$, which by distributivity is equivalent to $\mathsf{ext}\ U \cap \mathsf{ext}\ V = \mathsf{ext}(U \downarrow V)$, would bring to $a \lhd b \,\&\, a \lhd c \to a \lhd b \downarrow c$, which is trivial since $a \lhd b \,\&\, a \lhd c$ gives $a\,\varepsilon\, b \downarrow c$ by definition. In fact, the purpose of $\downarrow$-Right is exactly to express distributivity formally, and that is why we must start from $\mathsf{ext}\ U \cap \mathsf{ext}\ V = \mathsf{ext}(U \downarrow V)$.

In this way we have obtained yet another definition of formal topology, simply as a formal basic topology in which $\downarrow$-Right holds. To distinguish it from that given in Section 2.1, one could call it a *balanced formal topology*, because the difference is the presence of a binary positivity predicate and the absence of the condition of positivity (see Section 1.2).

As I hinted at in Section 1.2, the variety of possible definitions is a richness which one should not be afraid of. In fact, at this stage of development it is hard to see which one will become the standard one. The different requests on the positivity predicate $\ltimes$ seem to be the analogue of different separation principles in pointwise topology. Like in pointwise topology, it will take time to find out virtues and defects of each assumption.

Some of the advantages of the definition given above are already clearly visible. The first is that it has a solid structural motivation. In fact, the new predicate $\ltimes$ is the result of the isomorphism between concrete closed and formal closed subsets, and at the same time it is the symmetric of the interior operator int and the dual of the operator $\mathscr{A}$, that is of the cover $\lhd$. So $\ltimes$ seems to be exactly what is necessary to make the definition fully balanced. The second is that in this way it allows us to introduce a natural notion of formal closed subset; recall that a subset U is said to be formal closed if $\mathscr{J}U = U$ or equivalently if $a\,\varepsilon\, U \to a \ltimes U$. The third is that the richness of the structure allows to see that it is better to get rid of the condition called positivity (and study it as an extra assumption, if wished). In this way one can obtain both the theory of locales (or frames) and the previous version of formal topology as special cases. In fact, we say that $\mathscr{J}$ is improper when $\mathscr{J}U = \emptyset$ for any U; then one is left essentially only with the cover $\lhd$, which amounts to a predicative formulation of frames. We say that $\mathscr{J}$ is trivial when $\mathscr{J}S$ and $\emptyset$ are the only two formal closed subsets. One can prove constructively that $\mathscr{J}$ is trivial exactly when it satisfies $a\varepsilon\mathscr{J}U \to a\,\varepsilon\, H \,\&\, H \subseteq U$ for some monotone subset H (H is monotone if $a\,\varepsilon\, H \,\&\, a \lhd U \to H \between U$; when $\mathscr{J}$ is trivial, put $H \equiv \{a : a \ltimes S\}$). So a formal topology in the sense of Definition 1 is obtained as a special case by defining Pos to be H and by requiring the condition of positivity. Of course, a fourth advantage is that, as noticed with the definition in Section 2.1, some important examples of formal topologies fall under the new definition in a very natural way.

I expect also other advantages, or applications, to become visible after learning how the new expressive power—due to the presence of $\ltimes$—can be exploited. Before that, one has to adjust all the definitions and results of formal topology to take care also of the binary positivity predicate $\ltimes$. This does not look to be a routine task. As an example, I give here the new definition of formal point. Another example is given in Section 2.7.

The definition of formal point of a (balanced) formal topology $\mathcal{S} = (S, \lhd, \ltimes)$ is obtained as usual by considering the case in which $\mathcal{S}$ is presentable. So assume that $\mathcal{S}$ is the structure induced by a concrete space $(X, \Vdash, S)$ on the set S. The idea is first to describe the formal properties of a subset $\Diamond x$ traced on $\mathcal{S}$ by a concrete point x, and take them as abstract conditions for a subset $\alpha \subseteq S$ to be called a formal point (see also Section 3.2.2). Recalling that $\downarrow$, $\lhd$ and $\ltimes$ in the presentable case are *defined* by means of concrete points, we see that the properties we need are simply

$$\frac{x \Vdash a \quad x \Vdash b}{x \Vdash a \downarrow b}, \qquad \frac{x \Vdash a \quad a \lhd U}{x \Vdash U}, \qquad \frac{x \Vdash a \quad \Diamond x \subseteq U}{a \ltimes U}.$$

The first says that $(X, \Vdash, S)$ satisfies B2, the second and third are just a re-formulation of the definitions $a \lhd U \equiv \mathsf{ext}\ a \subseteq \mathsf{ext}\ U$ and $a \ltimes U \equiv \mathsf{ext}\ a \between \mathsf{rest}\ U$. We also add $\exists b(x \Vdash b)$, which corresponds to B1. Now we can transform such properties into properties of $\Diamond x$ by writing $a \,\varepsilon\, \Diamond x$ in place of $x \Vdash a$, and of course $\Diamond x \between U$ in place of $x \Vdash U$, and then take them as properties of an arbitrary $\alpha \subseteq S$. But if we now write $\alpha \Vdash a$ for $a \,\varepsilon\, \alpha$, we see that the definition we look for is obtained by literally replacing α for x in the properties above. So we have that $\alpha \subseteq S$ is a formal point if

α is inhabited: $\quad \alpha \between S$,

α is convergent: $\quad \dfrac{\alpha \Vdash a \quad \alpha \Vdash b}{\alpha \Vdash a \downarrow b}$,

α splits $\lhd$: $\quad \dfrac{\alpha \Vdash a \quad a \lhd U}{\alpha \Vdash U} \quad$ (where $\alpha \Vdash U \equiv \alpha \between U$),

α enters $\ltimes$: $\quad \dfrac{\alpha \Vdash a \quad \alpha \subseteq U}{a \ltimes U}$.

The condition that α splits $\lhd$ is actually redundant (in fact, it can be deduced from α enters Pos and compatibility), but I prefer to leave it explicit both to help intuition and to see that, when $\ltimes$ is trivial, the above definition gives back the definition of formal point previously given in [32].

As a last remark, note that the definition becomes much shorter in the notation with $\mathcal{A}$, $\mathcal{J}$ and $\between$. I leave it to readers to check that it is equivalent to $\alpha \between S$, $\alpha \between U \;\&\; \alpha \between V \to \alpha \between U \downarrow V$, $\alpha \between \mathcal{A}U \to \alpha \between U$, $\alpha \between U \;\&\; \alpha \subseteq F \to U \between \mathcal{J}F$.

2.5. Formal continuity and convergence

A notion of morphisms between formal (basic) topologies is introduced by following the same methodology which led us to the notion of formal (basic) topology. That is, we consider the notion of morphism between basic pairs, alias relation-pair, and we look for the properties which are enjoyed by its component on the formal side, with respect to formal open and formal closed subsets. These will be the properties we require to characterize morphisms between formal basic topologies.

It can be shown that a relation-pair (r, s) is equivalently presented by each of the following two properties, symmetric to those mentioned in Section 2.3:

$$s\Diamond D = \Diamond r D \quad \text{for any } D \subseteq X,$$

which means that s is *formal closed*, and r is a method to determine the formal closed subsets of T, which are the existential image of formal closed subsets of S along s;

$$s^{-*}\Box\, D = \Box\, r^{-*}D, \quad \text{for any } D \subseteq X,$$

which means that s^{-*} is *formal open*, and r^{-*} is a method to determine the formal open subsets, which are the universal image of formal open subsets of S along s.

When only the formal side is considered, the relations r and r^{-*} are lost, and the properties characterizing morphisms between formal basic topologies are then just the properties enjoyed by s. However, once r and r^{-*} are forgotten, it is no longer possible to prove the two conditions, that s is formal closed and that s^{-*} is formal open, to be equivalent to each other (two finite counterexamples are given in [18]). Hence both of them are required. Thus a morphism between two formal basic topologies $\mathscr{S} \equiv (S, \mathscr{A}, \mathscr{J})$ and $\mathscr{T} \equiv (T, \mathscr{B}, \mathscr{H})$ is a relation $s : S \to T$ such that *i.* s is formal closed, that is $U = \mathscr{J}U \to sU = \mathscr{H}s\,U$ and *ii.* s^{-*} is formal open, that is $U = \mathscr{A}U \to s^{-*}U = \mathscr{B}s^{-*}U$. We call it a *formal continuous relation*, and we denote it by $s : \mathscr{S} \to \mathscr{T}$. One can prove (see [18]) that, in the notation with $\ltimes$ and $\lhd$, the two conditions on s are equivalent to:

$$\frac{a\,s\,b \quad b \lhd V}{a \lhd s^{-}V}, \qquad \frac{a\,s\,b \quad a \ltimes s^{*}V}{b \ltimes V}.$$

Several other equivalent characterizations are also possible, see [18].

Given any formal basic topology $\mathscr{S}$, one can always define the image of $\mathscr{S}$ along any relation $s : S \to T$, by setting $s\mathscr{S} \equiv \mathscr{T}' \equiv (T, s^{-*}\mathscr{A}s^{-}, s\mathscr{J}s^{*})$. This is a formal basic topology in which formal open subsets are just the universal images of formal open subsets of $\mathscr{S}$, and dually formal closed subsets are just the existential images of formal closed subsets of $\mathscr{S}$. It is the coarsest formal basic topology which makes s a formal continuous relation.

Following this definition of image, it can happen that $\mathscr{S}$ satisfies $\downarrow$-Right, while its image $\mathscr{T}'$ does not. So the notion of formal continuous relation is not the right notion of morphism between formal topologies. As the notion of formal topology was obtained by describing axiomatically the formal side of a concrete space, that is a basic pair satisfying B1 and B2, now the correct definition of morphism between formal topologies is obtained by describing axiomatically the right component of a relation-pair which preserves the validity of B1 and B2.

So assume that $\mathscr{S}$ is the formal topology which is presented by a concrete space $\mathscr{X} = (X, \Vdash, S)$. One can see that the image of $\mathscr{S}$ along a relation $s : S \to T$ is the same thing as the formal basic topology presented by the basic pair $(X, s\circ \Vdash, T)$. Since $(s\circ \Vdash)^- = \mathsf{ext}\ s^-$, this satisfies B1 and B2 if $\mathsf{ext}\ s^-T = X$ and $\mathsf{ext}\ s^-b \cap \mathsf{ext}\ s^-d = \mathsf{ext}\ s^-(b{\downarrow}d)$, for any $b, d \in T$. But then, since $\mathscr{X}$ satisfies B1 and B2 (that is $\mathsf{ext}\ S = X$ and $\mathsf{ext}\ U \cap \mathsf{ext} V = \mathsf{ext}(U \downarrow V)$), these two equations are equivalent to $\mathsf{ext}\ s^-T = \mathsf{ext}\ S$ and $\mathsf{ext}(s^-b \downarrow s^-d) = \mathsf{ext}\ s^-(b \downarrow d)$, and hence finally, by the isomorphism $Sat(\mathscr{A}) \cong Red(\mathsf{int})$, also to $\mathscr{A}s^-T = \mathscr{A}S$ and $\mathscr{A}(s^-b \downarrow s^-d) = \mathscr{A}s^-(b \downarrow d)$, for any $b, d \in T$. In the notation with $\lhd$, these are equivalent to

$$\text{totality} \quad S \lhd s^-T \qquad \text{convergence} \quad s^-b \downarrow s^-d \lhd s^-(b \downarrow d).$$

So a morphism between formal topologies is defined to be a formal continuous relation which satisfies totality and convergence; it is called a *formal map*. Now one can easily check that the image $\mathscr{T}' = s\mathscr{S}$ of a formal topology $\mathscr{S}$ along a formal map is a formal topology too.

The notion of morphism between formal topologies presented in [32] is easily seen to be a special case of formal map. It is important to observe that the conditions of totality and of convergence are automatically satisfied by a relation s when it is the right component of a relation pair (f, s), where $f: X \to Y$ is a function and $(X, \Vdash, S)$, $(Y, \Vdash, T)$ are concrete spaces (the proof is left to readers). This shows that, when only functions are considered, the notion of continuity includes that of convergence.

The reason motivating the name of formal maps is that they induce functions between the formal spaces determined by the formal topologies; actually, they are the predicative way to present such maps. In fact, it is routine to check that whenever $s: \mathscr{S} \to \mathscr{T}$ is a formal map between formal topologies and α is a formal point of $\mathscr{S}$, then the image $s\alpha$ of the subset α along s is a formal point of $\mathscr{T}$. Hence s induces a map between $Pt(\mathscr{S})$ and $Pt(\mathscr{T})$.

A detailed treatment of formal continuous relations and of formal maps is given in [18].

2.6. The problem of definitions

I have already noticed several times that the method to obtain the definition of a formal notion is that of taking as formal axioms all the relevant properties which hold in the presentable case. It is now time to analyse this more carefully. The main problems are: what does it mean to take *all* properties? how can one be sure that there are no other? And in any case, which are the right axioms for $\lhd$ and $\ltimes$?

Only recently it has become clear to me that the answers depend both on the choice of the language (that is, of the primitives) *and* on the choice of the foundational theory. We now see how the different choices give different results, in particular on three specific questions: should closed subsets be uniquely determined by open subsets? should the cover be always assumed to be inductively generated? should one assume the condition of positivity $(\mathsf{Pos}(a) \to a \lhd U) \to a \lhd U$? I will argue that in the most basic approach the answer must be no to each question.

Assuming classical logic, as we have seen in Section 2.1, in any basic pair the equation $\mathsf{cl} = -\mathsf{int}-$ holds. By the same reason, on the formal side $\mathscr{J} = -\mathscr{A}-$. Moreover, classical logic guarantees compatibility of $-\mathscr{A}-$ with $\mathscr{A}$ to hold: $\mathscr{A}U \between -\mathscr{A} - V \leftrightarrow$

$U \between -\mathscr{A} - V$ is classically equivalent to $\mathscr{A}U \subseteq \mathscr{A} - V \leftrightarrow U \subseteq \mathscr{A} - V$, which is the characteristic property of closure operators. So our definition of formal basic topology boils down classically to that of a set S with a closure operator $\mathscr{A}$. In this sense, it is not visible in a classical foundation. Note, however, that adding the law of double negation on subsets $--U = U$ to our definition is not enough to make it trivial by forcing $\mathscr{J} = -\mathscr{A}-$ to hold: in fact, when $\mathscr{A}$ is the identity, any interior operator $\mathscr{J}$ is trivially compatible with it. This seems to mean that the structure of basic topologies has after all some stability which goes beyond foundations.

Allowing a quantification over subsets, like in topos theory, given a set X with an interior operator int, one can define closure as usual by quantifying on all open subsets. In fact, the very definition of cl in a basic pair, namely $x \varepsilon\ \mathsf{cl}\ D \equiv \forall a(x \Vdash a \rightarrow \mathsf{ext}\ a \between D)$, can be expressed by

$$x\,\varepsilon\,\mathrm{CL}(D) \equiv \forall E(x\,\varepsilon\,\mathsf{int}\ E \rightarrow \mathsf{int}\ E \between D).$$

One can check directly that such CL is indeed a closure operator compatible with int, and that actually it is the greatest of such operators. However, it is more instructive to note that impredicatively the collection of open subsets is actually a set, defined by $\{D \subseteq X:\ D = \mathsf{int}\ D\}$. Then the above definition of CL coincides with the definition of $\mathsf{cl} \equiv \mathsf{rest}\Diamond$ in the basic pair formed by X, the open subsets and $x \Vdash D \equiv x\,\varepsilon\,D$.

On the formal side, by symmetry one can always define a cover $\lhd^2$ impredicatively in terms of a positivity predicate $\ltimes$:

$$a \lhd^2 U \equiv \forall W(a \ltimes W \rightarrow U \ltimes W).$$

To get an intuitive grasp of this definition, one should compare it with the pointwise definition of cover $\forall x(x \Vdash a \rightarrow x \Vdash U)$, and recall that $a \ltimes W$ expresses formally the existence of a point in $\mathsf{ext}\ a \cap \mathsf{rest}\ W$.

So also in the impredicative case the full structure of basic topology would not be visible, since one can always choose the cover, and hence formal open subsets, to be uniquely determined by formal closed subsets. Moreover, when $\lhd^2$ is defined as above, the formal topology $(S, \lhd^2, \ltimes)$ is actually presentable (with X the set of formal closed subsets and $U \Vdash a \equiv a\,\varepsilon\,U$). So the reasons for introducing formal topologies in this context are not so compelling. It is still unknown to me whether, conversely, one can find a similar impredicative definition of the positivity predicate in terms of a given cover $\lhd$. In the special case of a trivial positivity predicate Pos which moreover satisfies positivity, this is well known, and the definition is $\mathsf{Pos}(a) \equiv \forall U(a \lhd U \rightarrow U \between S)$.

Also in a foundation based on the notion of computation, such as Martin-Löf's type theory, there are some good reasons for a less general notion of formal basic topology than that given here. In fact, because of the validity of the axiom of choice, the cover defined on the formal side of a basic pair is always inductively generated (this was remarked by Martin–Löf, a proof is in [10]). So it is natural to require inductive generation of the cover $\lhd$ as part of the definition. In this case, a positivity predicate is uniquely determined, and it is the greatest positivity predicate that is compatible with $\lhd$, which is defined by co-induction, as shown here in Section 2.7. In the same spirit, one can prove that for a relation between two such formal basic topologies to be a formal continuous relation it is enough that it respects the axioms. In this sense the two conditions defining formal continuous relations are no longer independent. It is to be recalled, however, that here too the request that $\ltimes$ be determined by $\lhd$ remains a choice, and is not a theorem (see the result by Valentini mentioned in Section 2.7).

The foundation we have been working with so far is essentially just intuitionistic many sorted logic. No assumption is made on the nature of sets; in particular, no axiom of choice and no powerset axiom are assumed to be valid. So there is no principle which allows to reconstruct that half of topology, dealing with existential statements and with closed subsets, in terms of the other half, namely universal statements and open subsets. This is why one can never forget either of them. The main conceptual advantage is that the resulting mathematics respects both the intuition of computation, which underlies the justification of the axiom of choice, and the intuition of some kind of continuity, by which one can sometimes be in the position of knowing a statement of the form $\forall x \exists y$ to hold also without having a function giving y in terms of x.

A deeper analysis and a precise formulation of such a basic foundation is still to be done. This should not be, however, an obstacle for doing mathematics. A firmer grasp will probably come after some more advanced technical development and some specific applications. In my opinion, however, just studying that part of topology which is compatible both with Martin–Löf's type theory and with topos theory is definitely worthwhile.

Since the language is not fully specified, the problem of completeness remains open: does the definition of formal basic topology gather *all* the properties induced on the formal side of a basic pair? This is an interesting logical problem.[13] In my opinion, however, rather than looking for a formal expression of existing definitions, it is more interesting to investigate whether working—even informally—on a different foundation can bring to a theory, such as the basic picture, with an *intrinsic* mathematical meaning. It seems to me that this is possible, and that the basic picture can be characterized structurally as the theory of what remains invariant under transfer along a (continuous) relation. I now briefly explain some technical facts which seem to support this expectation.

Since the whole basic picture begins with an analysis of images of subsets along a relation, a crucial step forward is to obtain a structural, or mathematical characterization of arbitrary relations and of images along them. To this aim, the language of categories is of great help. Thus we think of $\mathscr{P}X$ and $\mathscr{P}S$ as categories with the operators $\cap, \cup, \bigcup_{i\in I}, \bigcap_{i\in I}$ and with the predicates $\subseteq$ and $\between$. Then the operators on

[13] To solve this problem, one must deal with the specific problem of the property of positivity (see Section 1). It can be shown (see [37]) that positivity is equivalent to the equation $\mathscr{A}U = \mathscr{A}(U \cap \mathscr{J}S)$, which in the presentable case is valid simply by intuitionistic logic. Still, I believe it is better *not* to put it among the axioms. One reason is the fact that, as communicated to me by Coquand and Valentini, if the cover is finitary and positivity holds, then $\mathscr{J}S$, alias Pos, is decidable (so a statement in Section 4 of [32] is wrong, by the subtle mistake of assuming wrongly that the intersection of a finite subset with an arbitrary subset is again finite). This would mean that in general the compactification of a given formal topology would not be a formal topology. Another reason is that it is still to be checked whether positivity is respected by continuous relations. And finally note that in any case one can add positivity as an extra assumption and study the class of topologies in which it holds.

subsets, beginning with $\mathsf{ext}, \mathsf{rest}, \Diamond, \Box$, are best thought of as functors. We have seen that they form two pairs of adjoint functors, $\mathsf{ext} \dashv \Box$ and $\Diamond \dashv \mathsf{rest}$. But if we consider two arbitrary pairs of adjoint functors $F \dashv G$ and $F' \dashv G'$, with $F, G' : \mathscr{P}X \to \mathscr{P}S$ and $F', G : \mathscr{P}X \leftarrow \mathscr{P}S$, nothing is said about the fact that they are induced by the *same* relation, or better by a relation and by its converse.

It is well known that any adjunction $F \dashv G$ between $\mathscr{P}X$ and $\mathscr{P}S$ is induced by a relation between X and S. In fact, the left adjoint functor F respects unions, and hence one can define r: $X \to S$ by putting $x\, r\, a \equiv a\, \varepsilon\, F\{x\}$ and then obtain that $F(D) = \bigcup_{x\, \varepsilon\, D} F\{x\} \equiv \bigcup_{x\, \varepsilon\, D} r\, x \equiv r(D)$. So F is the same as the existential image along r. Since $r \dashv r^*$, by the uniqueness of adjoints $G = r^*$ holds. Assuming a second pair $F' \dashv G'$ of adjoint functors in the opposite direction tells us that there is a second relation $r' : X \leftarrow S$ such that $F' = r'$ and $G' = r'^*$. So now the problem is: which condition should one add to characterize abstractly the fact that $r' = r^-$? The answer is, a posteriori, incredibly simple: $r' = r^-$ holds if and only if the two existential functors F and F' are linked by

$$F(D) \between U \text{ if and only if } D \between F'(U) \quad \text{for any } D \subseteq X, U \subseteq S. \tag{3}$$

In fact, if $r' = r^-$ then (3) holds, because for an arbitrary r we have

$$r(D) \between U \text{ if and only if } D \between r^-(U) \quad \text{for any } D \subseteq X, U \subseteq S. \tag{4}$$

(this is easily checked by intuitionistic logic). Conversely, if $F = r$, $F' = r'$ and (3) holds, then for every $x \in X$ and $a \in S$ we have $x\, \varepsilon\, r'a$ iff $\{x\} \between r'a$ iff $r\, x \between \{a\}$ iff $\{x\} \between r^- a$ iff $x\, \varepsilon\, r^- a$, that is $r' = r^-$. I call (4) the *fundamental symmetry*, and say that two functors F, F' satisfying (3) are *symmetric*, written $F \dashv\vdash F'$.

It is thus proved that four functors $F, G' : \mathscr{P}X \to \mathscr{P}S$ and $F', G : \mathscr{P}X \leftarrow \mathscr{P}S$ are those induced by a relation $r : X \to S$, that is $F = r$, $G = r^*$, $F' = r^-$, $G' = r^{-*}$, if and only if they form two adjoint pairs $F \dashv G$, $F' \dashv G'$ and, moreover, F is symmetric to F', $F \dashv\vdash F'$. This I suggest to call a *symmetric pair of adjunctions*. This useful characterization is natural and simple only because of the presence of $\between$ (and actually, it was discovered only *after* the introduction of this notation). It is now possible to take it as a guideline to obtain formal definitions. One can now see that the first formal definition, namely that of formal basic topology, is just the result of transferring the structure of the power of the set X (including also $\between$) onto the set S through a symmetric pair of adjunctions. In fact, the compositions $G'F'$ and FG, acting on S, are well known to be a closure and an interior operator, respectively. Recalling that when the relation r is denoted by $\Vdash$ we use the notation $r = \Diamond$, $r^* = \mathsf{rest}$, $r^- = \mathsf{ext}$ and $r^{-*} = \Box$, these compositions are nothing but $\mathscr{J} \equiv \Diamond\mathsf{rest} = FG$ and $\mathscr{A} \equiv \Box\mathsf{ext} = G'F'$. We now can see further that compatibility is the result of transferring $\between$ from X to S. In fact, $G'F'U \between FGV$ if and only if $F'G'F'U \between GV$ (because $F \dashv\vdash F'$) if and only if $F'U \between GV$ (because $F'G'F' = F'$) if and only if $U \between FGV$ (again because $F \dashv\vdash F'$).

We can similarly see the definition of formal continuous relation as the result of transferring an arbitrary relation $r : X \to Y$ along two relations $\Vdash_1 : X \to S$ and $\Vdash_2 : Y \to T$. In fact, defining $s : S \to T$ by setting $s \equiv \Vdash_2 \circ r \circ \Vdash_1^-$, and considering the formal basic topologies induced on S and on T, we see immediately that s

is formal closed and that s^{-*} is formal open. Now the idea of invariance is that if we repeat the process, that is look for the structure which is preserved under transfer along a relation starting from a formal basic topology $\mathcal{S}$ rather than $\mathcal{P}X$ and from a formal continuous relation s rather than an arbitrary relation r, we should obtain again the same notions of formal basic topology and of formal continuous relation. This is still to be understood better. A remarkable result by Gebellato seems to be a good starting point. It says that the afore-mentioned characterization of arbitrary relations as symmetric pairs of adjunctions can be "topologized" and extended to a characterization of a formal continuous relation s: $\mathcal{S} \to \mathcal{T}$ as a symmetric pair of adjunctions between the lattices of open and of closed subsets in $\mathcal{S}$ and $\mathcal{T}$. See [19] for a precise statement and proof.

2.7. The dark side of the moon

The treatment of existential statements, or of statements of the form $\exists\forall$ like that in the definition of interior, is the dark side of the mathematical planet. They have usually been reduced either to a negation (as in classical logic, where $\exists$ is the same as $\neg\forall\neg$ and hence closed is the same as complement of open, see Section 2.1) or to an impredicative quantification (closure defined in terms of all open subsets). The main aim of the basic picture, and of formal topology developed on it, is the beginning of a more direct, positive exploration of that kind of information which is usually treated as negative. Only time and further work will tell whether the mathematics which is beginning to come out is interesting and with interesting new applications. My expectation is that it should find applications in those sciences in which a careful management of information seems important, like computer science, theoretical biology and perhaps quantum theory.* However, the task of a mathematician at this stage is still that of investigating the mathematical aspects, basing on internal criteria, such as structure or mathematical aesthetics.

Specifically, the aim is to develop a mathematics which keeps on the scene as primitive also the notions which are connected with existential quantifications. The introduction of the notation $\between$ for meet has this purpose; in fact, it allows to transform logical argumentations involving the existential quantifier into mathematical arguments involving $\between$, which are based on a spatial intuition. The first step is then to treat closed subsets as independent of open subsets. On the formal side, this brings us to the introduction of the positivity predicate $\ltimes$ besides the cover $\lhd$. The exploration of the dark side should now consist mainly in working out which are the right conditions which must be added to previous definitions and which take care of $\between$, closed subsets, $\ltimes$ and of notions connected with them. We have already seen one example, namely the definition of formal point. We now see another, more striking example, that is the generation of the positivity predicate $\ltimes$ by co-induction.

*Added in proof. This expectation has become true earlier than I expected: Peter Hancock and Pierre Hyvernat (in their recent paper "Interaction, computer science and formal topology") have shown that basic topologies have promising applications in the field of computer science.

It has been shown in [10] that the most general way to generate a formal cover on a set S is to start from a family of sets $I(a) set (a \in S)$ and a family of subsets $C(a,i) \subseteq S (a \in S, i \in I(a))$. The intuition is that $I(a)$ is a set of indexes for the covers of a, and that $C(a,i)$ is the cover of a with index i, taken as an axiom. Then a cover $\lhd$ (I mean, $\lhd$ reflexive and transitive) is generated inductively simply by the rules (see [10])

$$\frac{a \,\varepsilon\, U}{a \lhd U} \qquad \frac{i \in I(a) \quad C(a,i) \lhd U}{a \lhd U}. \tag{5}$$

The new idea now is to generate the largest predicate $\ltimes$ compatible with $\lhd$ by co-induction, that is by forcing compatibility to hold by successively taking away elements which do not satisfy it. Given that $\lhd$ is generated from axioms $a \lhd C(a,i)$, to force compatibility it is enough to consider this case. And of course one must also force co-reflexivity to hold. So the rules are

$$\frac{a \ltimes U}{a \,\varepsilon\, U}, \qquad \frac{a \ltimes U \quad i \in I(a)}{C(a,i) \ltimes U}.$$

The relation $\lhd$ is the minimal relation satisfying the rules written in (5). This means that for every subset U, the subset $\mathscr{A}U \equiv \{a \in S : a \lhd U\}$ is the least among the subsets P satisfying $U \subseteq P$ and $C(b,i) \subseteq P \rightarrow b \,\varepsilon\, P$ for any $b \in S$ and $i \in I(b)$. In other terms, the following principle of induction holds:

$$\frac{a \lhd U \quad U \subseteq P \quad \begin{array}{c}[i \in I(b), C(b,i) \subseteq P] \\ | \\ b \,\varepsilon\, P\end{array}}{a \,\varepsilon\, P}.$$

Dually, for every U the subset $\mathscr{J}U \equiv \{a \in S : a \ltimes U\}$ is the largest among the subsets Q such that $Q \subseteq U$ and $b \,\varepsilon\, Q \rightarrow C(b,i) \between Q$ for any $b \in S$ and $i \in I(b)$. So the following principle of co-induction holds:

$$\frac{a \,\varepsilon\, Q \quad Q \subseteq U \quad \begin{array}{c}[b \,\varepsilon\, Q, i \in I(b)] \\ | \\ C(b,i) \between Q\end{array}}{a \ltimes U}.$$

Using these two principles, it is possible to prove that $(S, \lhd, \ltimes)$ is indeed a formal basic topology. By combining this with the treatment of $\downarrow$-Right in [10], one can also easily generate balanced formal topologies. This shows at least that there is a wealth of examples for the new definitions.

Moreover, there is a wealth of examples also of formal continuous relations. In fact, assume that $\mathscr{S}$ is generated as above by I, C and that $\mathscr{T}$ is similarly generated by $J(b) set (b \in T)$ and $D(b,j) \subseteq T \, (b \in T, j \in J(b))$. Assume that $s : S \rightarrow T$ is any relation respecting the axioms, that is satisfying $s^{-}b \lhd_{\mathscr{S}} s^{-}D(b,j)$ for any $b \in T$ and $j \in J(b)$. Then one can prove by induction that s^{-*} is formal open and by co-induction that s is formal closed.

The idea of a co-inductive generation of $\ltimes$ first came to Martin–Löf, in July 1996 soon after several conversations by the author on the basic picture and in particular on the $\forall\exists$-$\exists\forall$ duality between open and closed subsets. A joint paper is in preparation, which will include also a game theoretic interpretation of $\lhd$ and $\ltimes$. Valentini has later shown that one can force $\ltimes$ to satisfy any given axioms, fully independently of the axioms for $\lhd$. This shows that there is a wealth of examples in which formal closed subsets are by no means determined by formal open ones.

Some other mathematical developments connected with $\between$ and $\ltimes$ are in progress. In particular, Sara Sadocco is working on an algebraization of the structure of $\mathscr{P}X$ in which $\between$ is taken as primitive, besides $\subseteq$.

3. Some principles, some reflections

The aim of this final section is to give an organic, though preliminary, exposition of the reflections of a general character—on the meaning of mathematics, of constructive mathematics in particular, and on the role of foundational assumptions in the specific case of topology—which have always accompanied my work in formal topology (and which subjectively cannot be separated from it).

In my opinion, one should start from the beginning, that is from general questions like: what is mathematics? what is its meaning? etc. My general attitude, which I call *dynamic constructivism*, can now be found in [36]. So I can here concentrate, in Section 3.1, on seven principles which are a bit closer to the mathematical practice and a consequence of the general philosophy. Section 3.2 contains comments which apply to the specific case of topology, and of the present approach in particular. They should help to understand the reasons of some mathematical choices made here.

I hope that the resulting survey will contribute to straighten an odd situation: in fact, I talk explicitly about matters which mathematicians tend to leave to philosophers, while philosophers treat them so generally that they produce very little interest among mathematicians. The reader I ideally address to is an open-minded mathematician.

3.1. The seven principles of dynamic constructivism

3.1.1. Genuine answers

I believe that the first principle of any scientist, and premiss to all other, should be that of giving nothing for true unless personally verified, a sort of cultural allergy to truth “by authority” and to any matter of faith. Thus one should have a genuine answer, with good arguments, to any question in and about mathematics, beginning with that one which is the simplest to put: what is mathematics?

In my opinion mathematics is best conceived as a human tool for human knowledge. So no God’s eye point of view is necessary, or convenient: also if God exists, it remains up to our responsibility to interpret what he/she says. An acceptable definition is then that mathematics is the study of abstract mental structures, related to counting, measure, grouping, shape, motions, vicinity, etc., and their applications. Whatever definition one prefers, mathematics is considered as the most reliable knowledge we can reach: one could even define it in this way.

The main reason to study constructive mathematics is in the end—in my opinion—simply that it is more reliable and allows to know more than classical mathematics; this is not reliable enough, since it betrays our intuition (see [36]).

3.1.2. Constructivism as awareness and modularity

It is well known that many kinds of constructivism have been proposed. Brouwer's intuitionism has appeared, in the scale of history, immediately after Cantor's exploration of the infinite, and it can be described as an answer to it. It can be characterized by the presence of mental constructions, which means that no actual infinite is possible, and hence by the rejection of the excluded third. In the dilemma of inward-outward reality, classical mathematics chooses outward reality, including a notion of truth as already given. It was Brouwer who first spoke openly about intuition (which etymologically comes from look inside) and internal reality, showing how it can be linked with a conception of mathematics.

Among the possible ways to avoid the paradoxes, Poincaré and others insisted on predicativism: a mathematical object (notably: a subset) cannot be defined in terms of the collection to which it belongs. Hence, one cannot quantify over all subsets to introduce a new subset; or positively, one can quantify only when a meaning can be given to $\forall$, $\exists$, hence surely at least when all the elements of the domain are generated by fixed rules.

Topos theory (on which locale theory is traditionally based) is intuitionistic but not predicative. Martin-Löf's type theory (since 1970) is an intuitionistic and predicative set theory, which includes logic via the "propositions-as-sets" interpretation, and solves the problems connected with Russell's type theory. The distinction between set and type is essential; all the elements of a set are generated by rules which can be specified in advance.

Which kind of constructivism should one choose? We first have to make it clear what we should mean by constructivism. We are lucky that today it is possible to appreciate the value of constructivity without any specific ideological measure: the preservation of intuition, or faithfulness to reality, has now become also preservation of information, or of computational content (in computers).

It is impossible to work in mathematics, intuitively and informally as usual, and in the same time keep all the information. If we keep too close to the machine level, there will be too many details, and hence too many complications. Some ideal notions are necessary.

So control cannot be the same as preservation of all the information. My general principle is that constructivism is not a static self-imposed limitation to full information, but rather awareness of which idealization has been made to build up an abstract concept, and to express it formally.

To simplify the complexity of raw data, one has to idealize, that is, to forget or destroy some data. Classical mathematics is simple, because it is based on a very strong idealization, or destruction, and with very little awareness. But also a rigid self-limitation, like when fixing a formal system once for all, means little awareness of destruction. Dynamic constructivism cannot be reduced to a formal system, but it must remain a cultural attitude: to be aware of idealizations (that is, of what is forgotten) and hence to know what one can obtain with certain tools and certain principles.

In practice, an important aim is to develop mathematics, as much as possible, in a basic, or minimal theory, which is so neuter that it preserves (and hence is compatible with) all kinds of intuitions with which we feed it. In particular, it must be open-ended, both in the notion of proposition and in that of truth of propositions, and hence it must be predicative and intuitionistic, respectively. In fact, since in a constructive approach a subset of a set X is the same as a propositional function over X, a quantification over subsets of X in an essential way (that is, to define a new entity) means considering the notion of proposition to be fully determined, and thus leaving no room for future developments. Similarly, the law of excluded middle leaves no room for future increase of knowledge, or truth.

I had chosen at first Martin–Löf's type theory as a foundation. Among the existing foundations, it is in my opinion with no doubt the most convincing among those dealing both with technical and with philosophical problems. In fact, it gives a clear meaning to all basic mathematical and logical notions, and in the same time it is a formal system very carefully designed to keep full control of information (and in fact it is also a computer language, see Section 3.2.1). Exactly because of this property, Martin–Löf's type theory enjoys a strong existence property: the witness which allows to prove an existential statement $(\exists x \in D)A(x)$ is encoded in a proof c of $(\exists x \in D)A(x)$ and can regained, *within the formal system*, by the projection functions giving $p(c) \in D$ and $q(c) \in A(p(c))$. So the so-called axiom of choice is validated, actually, it is rigorously provable.

Martin-Löf's type theory seems thus perfect to deal with computations. However it is well known that the axiom of choice is constructively incompatible with powersets (see footnote 2). But it is a fact that in the development of formal topology the axiom of choice is used rarely enough that one can leave it out, and assume it only when necessary, and then with explicit mention. It is certainly never used in all the results mentioned in Section 2 here.

As regards the foundation as a formal system, this means that one has to abandon the proposition-as-set interpretation which is usually at the base of Martin-Löf's type theory, and give intuitionistic logic on propositions, with its usual inference rules, separately from sets. In this way one can keep the notion of set exactly as in Martin-Löf's type theory, and have a notion of proposition which does not satisfy the strong existence property (note: the existence property continues to hold, it is just that to obtain the witness one has to pass through the metalanguage). It is not my most urgent aim to formalize rigorously such a theory, but rather to see where it can bring us in the development of mathematics (see Section 3.1.4).

Such a foundation is fully compatible with the addition of quantifications over all subsets, and thus also with a direct geometric intuition, such as that behind topos theory or choice sequences. So most of the results of formal topology (that is, as long as the axiom of choice is not used) hold in an arbitrary topos. Conversely, a considerable portion of locale theory (by which I mean pointfree topology developed over the foundation of topos theory) is not possible predicatively, and hence is absent from formal topology.

There are at least four good reasons which make well worth the trouble of developing a piece of mathematics, like formal topology, over a "minimalist" foundation such as the one described above. The first is that it works, in the sense that it is sufficient to express topological notions and to work with them. The second is that it allows to see that the real basis of topology is explicitly that of symmetry and logical duality, rather than implicitly that of some notion of set, which is bound to often silent philosophical assumptions. The third reason is to begin a modular approach to the development of mathematics and the study of its foundations. Starting from a minimal foundational theory and developing mathematics over it, allows to analyse which parts and which peculiarities of mathematics depend on further foundational assumptions. This, rather than the choice of one foundational theory, should be in my opinion the study of foundations. In a certain sense, all means, that is all assumptions are allowed, also if beyond the basic type theory, as long as it remains clear when they are active. In fact, the rejection of a principle is not due to some kind of moralistic contempt or of ideological attitude, but rather seen as a method to keep some positive general features of mathematics as developed in the basic theory. Then it should be clear that there is nothing wrong, even for a convinced constructivist, to speak about some classical or impredicative results, as long as no confusion is made. In fact, it is often useful to be able to look at things from a certain distance and thus to put an upper bound to what one can hope to prove constructively. The fourth reason is that the actual development of mathematics in a "weak" theory helps to extend the territory of mathematics itself (see Section 3.1.4).

3.1.3. Forget safely

Assume that constructivism involves a strict control of information and, in particular, awareness of the information which has been forgotten, or destroyed, to be able to obtain a certain idealized concept. Then the idea is to develop mathematics directly treasuring constructivity, that is, being very careful when throwing away some information, in case it is not possible to restore it. This is the reason why a constructive treatment carries on more details than the classical one. The computational content is kept along the way, rather than put on top of a finished non-constructive work, when it may be too late. The typical example is that of the existential quantifier $\exists$. There is no way to prove constructively an existential statement except by having a witness or by a previous existential quantifier (with its witness). This is why $\exists$ is kept distinct from $\neg\forall\neg$. If $\exists$ is identified with $\neg\forall\neg$, the information about the witness is soon lost in an irreversible way. [14] So the choice for such identification, which might look as the choice for a "stronger" logical principle, actually means that the witness information is considered to be irrelevant.

[14] A nice example, from everyday life, is when, contrary to the rules of a serious craftsman, an antique piece of furniture is restored by cleaning it with sandpaper. One so abstracts from "dirt", but in this way its patina is lost and its age is forgotten, and hence the commercial value destroyed, in an irreversible way!

If awareness of destruction is the aim, then the ideal situation is to forget only when it is safe, that is, forget only that information which can be restored when needed, maybe by passing through the metalanguage (that is, for instance, by inspecting the proof of a certain statement). It is possible to give a mathematical form to this principle, which I call the forget-restore principle: an ideal notion can be introduced, by forgetting some information, only if it is effectively conservative, that is, if it can be reduced to the underlying type theory in such a way that also proof-terms, which express the computational content, can be restored.

The formal setup of Martin-Löf's type theory works perfectly well for this purpose. In fact, it provides us with a total control of information, and hence it is easier to keep track of what is forgotten. Moreover, type theory is known to be correct, either by direct meaning explanations or by normalization arguments. So to develop constructive mathematics it is enough to enrich it with some abstract "utilities", beginning with a notation closer to the usual style of mathematicians. Purging mathematics from all what does not strictly fall into the formalism of type theory would sacrifice human intuition, and in the end it would rightly be felt as a form of penance. Of course, any abstract tool which is introduced must preserve not only consistency, but also constructivity, and thus it must obey the forget-restore principle explained above. This has been done, for instance, for the notion of subset, see Section 1.3.1.

This also gives a new meaning—the only constructively possible, in my opinion—to Hilbert's program. One could call it the humble Hilbert program, because rather than trying to justify *all* of mathematics in one shot, the goal is to do it bit by bit, and add each safe bit to the basic theory, which is known to be safe.

It is certainly convenient to do this in a modular way, that is in such a way that different tools can be put together at will, one on top of the other. For this purpose, preservation of predicativity seems to be necessary. In fact, quantifying over all subsets to introduce a new tool means that the notion of subset is blocked, and this can be incompatible with the addition of a second tool, like a new set-constructor.

For some other comments, see [39] and Section 3.2.1. See also [46] for another example following the forget-restore principle.

3.1.4. New foundations must give new mathematics

Reality is too complex, chaotic even. It is a need of ours to organize it in some way, looking for patterns, regularities, abstract concepts. This is anyway something we impose on reality, and to obtain it we have to forget details, that is idealize. Mathematics is an important tool for this general aim. The different foundations correspond to different choices of how reality is simplified.

Classical mathematics corresponds to the strongest idealization: all what is consistent is assumed to exist by itself. All propositions, all subsets, all objects of mathematics live in a single and static world, which has been and will always be as it is now. The task of mathematicians is just that of discovering what is already there and true in itself.

There must be a good reason to push one to abandon such a simple view. It is a common opinion that the purpose of constructive mathematics is to repeat constructively as much as possible of classical mathematics. If this were the only aim, constructive mathematics would soon become boring (I believe that this view is actually one of the reasons explaining why so few mathematicians work in constructive mathematics, see Section 3.2.5). On the contrary, I believe that the real motivation for choosing a different foundation is that it leads to some new mathematics, which would not be possible otherwise.

For instance, choosing category theory and intuitionistic logic has brought to the novelty of topos theory. The novelty of mathematics over a predicative foundation is usually measured in terms of computation. It is well known that mathematics developed predicatively, in particular if within type theory, is automatically formalizable in a computer system, and hence can for instance be checked mechanically (see Section 3.2.1). This is sometimes considered to be the reason for developing predicative mathematics. Certainly, it is extremely important and interesting, but according to my personal taste not always so exciting intellectually. It would not be worthwhile for me to take pains to avoid excluded third and powersets if the aim were only that of confirming what already exists. The purpose of a new foundation must be that of finding some new mathematics, that is new ways to organize reality into conceptual structures.

The crucial step for this change of attitude is to perceive the refusal of "powerful" principles not as a more or less meritorious deprivation, but rather as a refinement of mathematical instruments, and hence a source of richness. After all, allowing oneself powerful principles for any purpose and in any situation is not so different from using a butcher knife also when a surgeon knife would be more suitable. A less brutal metaphor is that of wearing always a pair of coloured glasses, so that not all colours can be distinguished.

A "weak" foundation allows for distinctions which are impossible otherwise. It is known that intuitionistic logic is more refined than classical logic, in that it keeps logical constants distinct. One begins to see that the same holds for a predicative foundation, which keeps the distinction between a set and its power, otherwise lost in an irreversible way (see also Section 3.2.2).

The method of reducing assumptions to obtain deep and general structures is well known in mathematics, at least since the beginning of abstract algebra. The same happens with foundations: trying to express a concept in a language with finer distinctions, like a weak foundation, often produces deeper understanding and new structures, that is new mathematics. To see this happen in practice, it is necessary to put that foundation at work and actually *use* it to do mathematics. Proving *meta*mathematically that something is possible does not help much to discover new things. Conversely, the awareness acquired by users often improves the understanding of the role of some foundational principles. So I see the study of foundations as a dynamic process which is active in both directions, rather than a justification a posteriori of something taken as given.

Using type theory, and actually only the small fragment coinciding essentially with many sorted intuitionistic logic, to develop topology has produced the new mathematics which I call the basic picture. The materials on which it is based are (no plastic or sand, nor any ideological assumption, but) the hardest a logician and a mathematician can expect, namely very elementary logical dualities and geometrical symmetry. The duality underlying the notions of closed and open gives a strong motivation for a study of topology, both pointwise and pointfree, in which the notion of closed is primitive like that of open. The symmetry between the concrete and the formal side allows to embed both the pointwise and the pointfree approach to topology in a unified framework.

All of this would probably never have seen the light in a different foundation. It is the use of a minimalist foundation which forces one to find new explanations, and hence new structures. As discussed in Section 2.6, classical logic would bring to the identification of closed with complement of open, and thus their duality would collapse to complementation. Impredicativity would make the set on the formal side always definable as the set of open subsets. Thus in any case the basic picture would remain just a consistent but funny option, and thus it would escape to our attention (as it has been, as a matter of facts). One thus can see that in a precise sense stronger foundations begin with being less constructive, and end with being more destructive.

3.1.5. Compatibility

At the origins of constructive mathematics, it was natural for Brouwer to oppose to classical mathematics. He had to break the ice and conquer attention. Perhaps for this reason, he introduced some principles which are incompatible with classical mathematics. These assumptions are actually not necessary to develop constructive mathematics. For this reason (and for the birth of computers) constructive mathematics today is less ideological.

As a principle, constructive mathematics should be developed while taking care that all definitions and proofs are compatible with a classical reading. This is possible by using a toolbox of notions and notations which allows one to use common mathematical language and still guarantees formalizability in the basic type theory.

pt The main purpose of compatibility with a classical reading is communication between different traditions. So I also believe that some specific words introduced by Brouwer, like spread, species, etc., should now be replaced by those common among mathematicians, like tree, set, etc. It is a different reading which must give the different interpretations (in the same way as written Chinese is read in different ways).

Full compatibility should avoid miserable fights, as at the times of Hilbert and Brouwer (assuming we don't have the same difficult character). The matter should not be a choice of side in the battle field, but simply of the kind of quality of information one is interested in.

Compatibility should encourage communication also at the level of contents. On one hand, high idealization is often useful to constructivists, either to get inspiration or to foresee what cannot be expected, that is to get negative information. On the other hand, classicists can appreciate at least the technical improvements often accompanying a constructive formulation. By the principle of tolerance, they are left free to destroy information *and* structure, if this remains their will.

3.1.6. Importance of definitions

The classical mathematician tends to believe that doing mathematics mainly coincides with producing proofs. In a constructive approach, choosing the right definitions is also important. Actually, choosing definitions and testing them to be correct is the main part of the work. To choose the right balance between simple idealization and the amount of information to be kept is at least as difficult as to prove a theorem in a classical approach. But on the other hand, adopting good constructive definitions usually has the effect that proofs become much easier, reasonable and perspicuous. And so it must be: if we want constructive mathematics to reflect our intuition better, then a proof should not come so much as a surprise.

Two principles have turned out to be helpful when looking for good constructive definitions. Since, as a matter of facts, the classical definitions exist and are well known, one can start from them and consider them as a first, strong idealization. The phenomenon of the intuitionistic "diffraction" of classical definitions is well known: many classically equivalent characterizations of the same notion are no longer equivalent in intuitionistic logic, and thus the same classical notion may correspond to several different intuitionistic notions. Since the law of excluded third identifies existential statements with negation of universal statements, to obtain the intuitionistic version one has to understand which are the positive existential statements which have been confused with negative ones. With predicativity another similar phenomenon becomes observable. In fact, because of the powerset axioms, in an impredicative approach sets are confused with collections, and the information that something is a set is lost. So to add predicativity one has to restore and keep on the scene those sets, or other entities, which have been forgotten on the assumption that they can be reconstructed by an impredicative principle.

In both cases something has been left out. Thus one should reverse the perspective: it is not a matter of translating a classical definition which is assumed to be right. Rather, the first principle is to assume that the constructive definition to be found is the correct one, and that a less constructive one has been obtained by forgetting unsafely some of its components. In practice, the difficult and creative part is to discover and put back what the standard definition has left out. Type theory is very suitable for this purpose, since a predicative formulation is often nothing but the expression of a given notion in the formalism of type theory (this is somehow analogous to the fact that defining something in the categorical terms of objects and arrows often is the same as finding its right structure).

The second principle is to recall that compatibility (see Section 3.1.5) must include also definitions. That is, the constructive definition, whenever possible, should be readable, exactly with the same words, by classical mathematicians and give an equivalent to their definition (of course, their reading of "set" and "exists" will be different: this is their taste!).

All these general remarks become very well visible when specified to topology. First, in the standard approach one assumes the family of open subsets to form a set (rather than a subcollection of the collection of all subsets). In this way the information on how it has been presented, for instance by means of a function from a second set as in a concrete space or by a direct inductive generation of a formal topology, is lost completely. By these reasons the standard definition is split into three different constructive notions: concrete space, formal topology, formal space (see Section 3.2.2 for more on this). Secondly, since the law of excluded third identifies closed subsets with complements of open subsets, some creativity was necessary to realize that the real link between open and closed is a rich logical duality, and not just complementation.

The basic picture somehow "proves" the correctness of the constructive definitions by showing in detail on what ground structure they are based. And yet all is perfectly readable also for a classical mathematician (who will anyway wonder about the reason for so many useless distinctions and additions).

3.1.7. Harmony, ecology and aesthetics

Mathematics is one of the achievements of which humanity is rightly proud. Together with arts, science, ethics, etc., it is a part of culture, and this can be seen as a continuation of natural evolution within the human species. Creating and exploring the world of mathematics can be a fascinating and exciting experience. It becomes also a pleasure and a real enrichment of mental life if this is done in full harmony with nature and with the nature of human mind. Mathematical knowledge should be free of any supernatural interference, that is will, prejudice, dogma, expectation or fear. This cultural attitude is in my opinion the deepest motivation for constructive mathematics. One could even say that it *defines* when mathematics is constructive. Brouwer was well aware of this, but he was apparently ahead of his times.

When a notion of knowledge and truth as harmony with nature begins to be a part of one's view of the world, then one begins to realize that, contrary to a common belief, it is the classical approach which is less free, since it forces reality into unnatural principles. The sharp division of the world between good-truth and evil-falsity appears as the mathematical continuation of a childish wish of omnipotence. To justify it, one is bound to adhere to some kind of faith, like the existence of platonic ideas, or even worse to divide the self, that is split body from soul, form from content, and retain only the shadow of truth which is materially perceivable in formulae.

A respectful attitude in doing mathematics is something one has to learn, or actually to teach oneself, since very few of us have been educated to it (just as very few have been educated to a respectful use of resources). It may look at the beginning that giving up supposedly strong principles is like self-inflicting a punishment or even a mutilation; even some followers of constructivism have felt this way. Getting rid of dogmas, breaking the rigidity due to fears and accepting that the world continues to be what it is independently of our personal absolute certainties: all of this certainly is a cost in psychic terms. But only in this way one can learn to see things as they are, and hence reach a higher stage of awareness and knowledge. And since mathematics is mostly a mental organization of abstract concepts, to reach a stage in which it becomes a fully natural, meaningful and convincing activity, one has to strive for and keep a strong internal harmony between the different aspects of mathematical thought, namely spatial intuition, computation and logical deduction. This is the therapy I suggest to get cured from the schizophrenia in contemporary mathematics pointed out by Bishop [5] and reach what he calls "integrity". In other words, it is the recipe I propose to build a world of mathematics which is not strange to ourselves and hence in which it is a pleasure to live and work.

I believe that this is really possible and not just wishful thinking. I am not proposing a dramatic revolution, the repetition of a putsch or pure mystical contemplation with no practical value; this fears should now be only a bad memory from the past. The aim is not to "hasten the inevitable day when constructive mathematics will be the accepted *norm*" ([4], my italic), but the day in which all mathematicians will be free and free to choose constructive mathematics, because they will be fully aware of the right foundation for each purpose. If anything is at all inevitable, this to my eyes is the day when it will be realized (perhaps after deeper use of computers) that the classical foundation is not good for all purposes. The fact that so many still believe it is, remains to my mind one of the unsolvable puzzles of our culture. In fact, even if one disregards positive arguments in favour of alternatives, one should at least realize that depending on a single foundation in our mental life is as dangerous as depending on a single source of energy (like oil) in our material life.

What I propose is simply to reach a more balanced view and to begin in practice by putting more energy in the development of alternatives, that is in the direct development of mathematics over different foundations. It makes no sense to charge any kind of constructive mathematics of providing no really viable alternative, until the amount of work which is put on it remains marginal; this is as silly as rejecting something like electric or hydrogen cars, comparing them with a Ferrari before they are properly developed *and* forgetting that their purpose is different.

The analogy with ecology is so strict and illuminating that it can be taken as a guide. In this terms, my general proposal becomes simply common sense: do *now* all what is possible to preserve integrity of the mathematical environment, in all its forms, and to improve the quality of life in it. So the vitality of mathematics should not be measured only by the number of theorems produced. To the contrary, just as producing more cars brings to more chaos and traffic jams, a blind overproduction of theorems at any cost has the negative effect that their meaning and significance are lost. It is no longer clear what the real progress is. A good discipline to recover meaning is the ecological principle of producing theorems with a minimal use of conceptual resources, that is, in the weakest possible foundation. Most mathematicians, while being careful in choosing a minimal stock of axioms for their theories, have little or no scruple about axioms in their meta-theories. The enormous waste of foundational assumptions is made worse by the little knowledge of their impact on the environment and on the theories themselves. For instance, little is known about the pollution they produce, in the sense of the collapse of different notions and hence the death of important distinctions of meaning. One observable negative effect is the difficulty in building safe mathematics and hence safe computer programs.

Similarly, one should avoid waste of ad hoc definitions, which are easy to produce but difficult to recycle, like plastic products. They are often broken after short use, but remain there and overpopulate our mental space.

In the world of mathematics, the various kinds of environments are just the different kinds of mathematics provided by various foundations. As in the biosphere, plurality is an essential ingredient of life and hence a source of richness. To save biodiversity one has to take care and keep each foundation alive and in good health, by exploring and actively developing the specific mathematics it gives rise to. Here there is a lot of mathematical work waiting for somebody to do it. To be able to see it, one must first simply abandon the strong principles without fears, just like to be able to see alternatives to cars one must first leave one's own car in the garage.

Exploring the world of mathematics with a weak foundation is like travelling on foot or riding a bicycle, rather than by car or plane. The pleasure is not that of possessing as many places as possible by passing through them, but of getting familiar with the landscape and harmonious with nature. In this way, one can observe many facts which otherwise, due to speed or distance, would remain unnoticed. This means acquiring a kind of knowledge which is impossible otherwise. Moreover, a considerable side effect is that of avoiding crowds (up to now, at least).

Taking care of aesthetics is a good antidote against the uglyness of plastic definitions, the absurdities of wasting the foundational resources, and the will of power over the environment. In fact, it seems to me that our sense of beauty is a deep sign of harmony between our internal world and the perceivable world. In mathematics, it is usually called elegant, or even beautiful, what pleases our need of appropriate mental structures. I believe that mathematicians with some maturity or sensitivity can understand what I mean without further explanations; or at least, they have certainly experienced it sometimes. This is why in my own exploration I have often let myself be inspired by my aesthetic discernment.

Certainly the inevitable day should be hastened when humanity will realize that its present behaviour is destroying the world. Unfortunately, this has already happened locally in the past; with globalization and increase of power, there is now much more at stake. Luckily, this is not a direct effect of careless mathematics, but perhaps it is precisely from mathematics that one can begin to develop a new, more respectful culture and view of the world. Leaving any dream of absolute virtual power and going out of the world they have built for this purpose, mathematicians can become aware and responsible of their limited but real power. The new conception of constructivism is not a doctrine which is born out of new dogmas or restrictions. It is just the wish of a real, trustable knowledge in harmony with nature, inside and outside our brains, and that is in the end just love and respect of oneself and the others.

3.2. More concrete points in topology

3.2.1. Mathematics and computers

One should give for granted, I believe, that the role of computers in mathematics is going to increase in the future. Computers will be used not only as calculators, but also as assistants in the task of developing mathematics and checking it to be correct and safe. To this aim, mathematics must be formalized in a programming language.

One of the most intriguing aspects of Martin-Löf's type theory is that two important motivations, namely the foundation of constructive mathematics and of computing science, converge to the same result. In fact, type theory has shown that the detailed formalization of an intuitionistic and predicative set theory becomes *ipso facto* the specification of a high level programming language (see [27,26,30]).

The same holds for mathematics, and not only its foundation. To obtain a piece of mathematics which can be implemented in a machine, the most effective method is just to develop it directly within type theory. In this way the philosophical arguments in favour of constructive mathematics converge with (or are replaced by, according to taste) a practical, non-ideological motivation.

To be able to express a specific definition or theorem in type theory one often has to reduce it to its deepest constituents and understand it so well that it is no wonder that the computational meaning can come to surface in the form of a computer program. In practice, however, the full formalism of type theory is so detailed that a piece of mathematics written with no abbreviations becomes unreadable to the standard mathematicians. This is not a defect of type theory, since it is designed precisely for a careful preservation of all the information, even that which is redundant in usual mathematics; actually, type theory is totally trustable because of this. But if the aim is to provide mathematicians with an assistant to *their* activity, it is a task of the assistant to understand the language of mathematicians. This is the aim of building up a toolbox for type theory, that is an interface providing type theory with notions and notations which on one hand allow the mathematicians to work in their usual way, but on the other hand guarantee that the computational content is not lost (see Section 3.1.3). Even if the language is standard, methods of proof remain fully intuitionistic and predicative. So the price that mathematicians have to pay is the development of a new mathematical intuition, and this requires time and education; note however that this holds also for the intuition underlying classical or impredicative mathematics, even if a posteriori it is given for granted. Another price to pay is the development of the toolbox and of its implementation. It seems just reasonable to try and make each tool as independent as possible of specific implementations of type theory (since these change quickly, while mathematics should remain more stable). For instance, the whole theory of subsets depends only on two simple conditions (see Section 1.3.1), whose implementation can change at will without affecting the tool of subsets.

The presence of a toolbox for type theory makes the interaction between mathematics and computer science simpler and more intense at the same time. As long as they use only the toolbox, mathematicians can continue their job with no worry about making their results closer to formalization in type theory, to please the person who is going to do it. They are responsible, so to say, from toolbox up. And still they know that their results can be mechanically checked and proved to be safe. Computer scientists should not worry about formalizing all what mathematicians produce, but should take care only of what is necessary to implement the toolbox. They are responsible from toolbox down.

If the aim of formalizing mathematics is to mechanically check its safety, some kind of tool is not only useful, but also theoretically *necessary*. In fact, what would be the gain in safety if a vast amount of human work is necessary to bring mathematics inside a formalism? How is *that* checked?

The development of a rich toolbox should therefore be of general interest. The single tool of subsets has been enough, up to now, for the purpose of developing formal topology. So, in particular, all the mathematical development in this paper really *is* automatically formalizable in type theory, hence mechanically checkable and (most probably) safe.

3.2.2. *Do points exist?*

The predicative approach to topology leads in a natural way to the consideration of opens as given primitively, i.e. to the so called pointfree or formal approach (see [25,21]; the name is due to the fact that an open subset is only formally so, from the traditional perspective at least, since it does not consist in a subset of points). Actually, as I will show below, predicative topology *must* contain the formal approach. For this reason, it is sometimes believed that it must *coincide* with it.

Beginning in particular with the emergence of the basic picture, I have been developing a change in conceptual understanding, which now looks to me as the most reasonable and open minded. One can summarize it in three points as follows:

(a) Keep concrete points, when there are some. It is a task of the predicative foundation and its users to distinguish concrete points from formal points, that is concrete spaces from formal spaces.

(b) Reappraisal of the pointfree approach not as a substitute of the pointwise one, but as interesting in its own right.

(c) Develop an intuitionistic and predicative topology as a primitive, and not as a way to recover as much as possible of classical topology. This must then lead to some new mathematics (see Section 3.1.4).

I now try to explain these points in all details, but not singularly since they are strictly interconnected.

The whole basic picture was born by analysing as deeply as possible the definition of formal topology given in [32], with the aim of improving on it. The idea is to study in detail the case of a topological space X in which the topology $\mathcal{O}X$ can be concretely presented by means of a base of open subsets indexed by a second set S. To this aim, as with topological systems introduced by Steve Vickers in [48], one has to keep both the points and the structure of opens in one framework, to be able to formulate better their mutual relationship. A topological system is a triple $(X, \Vdash, \mathscr{L})$ where X is a *set* of points, $\mathscr{L}$ is a frame (or locale) with a set S of elements and $\Vdash$ is a relation between X and S binding points with the structure of $\mathscr{L}$ in the expected way (that is, $x \Vdash 1$, $x \Vdash a \wedge b$ if and only if $x \Vdash a$ and $x \Vdash b$, $x \Vdash \bigvee_{i\in I} b_i$ if and only if there exists $i \in I$ such that $x \Vdash b_i$). This definition is meant to include the case in which X is formed by all completely prime filters, alias formal points of $\mathscr{L}$. Indeed, so it is, impredicatively; predicatively, such X is not a set, since it is a collection of subsets of S. A predicative version of Vickers' definition could then be: a triple $(X, \Vdash, \mathscr{S})$ where X is a *collection* of points, $\mathscr{S}$ is a formal topology and $\Vdash$ binds X with $\mathscr{S}$ in the expected way (that is, as in the definition of formal point, see Section 2.4). This definition would include formal spaces, that is triples $(Pt(\mathscr{S}), \Vdash, \mathscr{S})$ where $Pt(\mathscr{S})$ is the collection of formal points of $\mathscr{S}$, and $\Vdash$ is of course reverse membership. It would however not allow an easy analysis, since quantifications over $Pt(\mathscr{S})$ are not meaningful predicatively. So one has to take courage and restrict to the case in which X is a set also predicatively. In this way formal spaces are ruled out, but this is compensated by other important advantages. In fact, one can always define a formal cover $\lhd$ on S by quantifying over points in X (which is now possible, since X is a set) and by putting $a \lhd U \equiv (\forall x \in X)(x \Vdash a \rightarrow (\exists b \,\varepsilon\, U)(x \Vdash b))$. Of course one can also define Pos by putting $\mathsf{Pos}(a) \equiv (\exists x \in X)(x \Vdash a)$. Moreover, one can get rid of 1 and the operation $\cdot$ by introducing $\downarrow$ (as explained in Section 2.1) and thus in the end obtain a formal topology $(S, \lhd, \mathsf{Pos})$, as defined in Section 2.1. So the assumption that S comes equipped with the structure of a frame (or of formal topology) can be dispensed with, and one is left with the notion of concrete space (see Sections 1.1 and 2.1; the sign $\Vdash$ remains as a trace of the link with [48]).

An extra benefit, which was not expected, is that the expression of the fact that S gives a base for $\mathcal{O}X$ takes the form of two conditions, called B1 and B2, to be put on top of the relation $\Vdash$. Then one can realize that the usual *definition* of open and of closed subsets of X is possible also when B1 and B2 are not assumed to hold. So one can further reduce to the most simple structure consisting of the sets X, S and the now arbitrary relation $\Vdash$ between them; the properties B1 and B2 can be added at will.

This is the conceptual path which brings to basic pairs, and to see them as a generalization of concrete spaces (which impredicatively are the same as topological spaces). Now this path can be reversed and one can use the ground notion of basic pair as a starting point to get a unified and deeper perspective on topology, formal topology in particular. In fact, a basic pair is the most elementary structure in which topological notions can be anatomized and reduced to their deepest essence. By analysing basic pairs, one discovers that the notions of open and closed are linked by a logical duality, and that the concrete side (of concrete points) and the formal side (of formal basic neighbourhoods, or observables) are linked by symmetry (see Section 2.2). The geometrical and topological intuition is thus supported by a solid structure, which one can then extend to less elementary situations. In practice, duality and symmetry are kept as guiding principles for a correct further development.

The examples of basic pairs, and hence also of concrete spaces, are not so many predicatively, and perhaps also not so interesting. The reason for introducing formal topologies is precisely to obtain a more general approach. One can then generalize also the notion of point by introducing formal points over a formal topology and hence finally also that of concrete space by introducing formal spaces. Thus the first task is that of finding a correct definition of formal topology; this is the main purpose of the detailed analysis of the case in which points *do* form a set. In fact, the definition of formal topology is obtained by abstraction of the structure induced on the formal side of a concrete space. In other words, let us call this structure a (concretely) presentable formal topology; then we define formal topologies abstractly by requiring as axioms those properties which are valid in the subclass of presentable formal topologies. It is clear that in this way the importance of the notion of formal topology is *not* diminished, and that the number of spaces which can be presented concretely is *not* increased (certainly not by subtly weakening the conception of constructivity). What is increased is the epistemological value which is given to the special class of presentable formal topologies, that is in the end concrete spaces. They are seen as a lucky case in which one can analyse with all comforts the links between (concrete) points and (formal) opens, as well as the effects they produce on the formal side, which are later to be taken as axioms. And this idea would continue to work well even if the only interesting concrete spaces were the finite ones.

The discoveries reached through the analysis of the special case in which concrete points do form a set result in some conceptual improvements on formal notions. The first is that one can consider basic pairs, rather than concrete spaces, and hence introduce the new notion of formal basic topology (see Section 2.4), which is obtained by abstraction of the structure induced on the formal side of a basic pair. As a concrete space is nothing but a basic pair in which (B1 and) B2 hold, so the definition of formal topology is now obtained as a special case by adding as axiom the property which is induced by B2 on the formal side, and this is $\downarrow$-Right (see Section 2.4). This is the beginning of a generalization of formal topology, which is a part of what I called the basic picture. Another improvement is that one can rely on the symmetry between the concrete and the formal side of a basic pair, and hence transfer the duality between concrete open and closed subsets also to the formal side; this means that it becomes natural to introduce a new relation, namely the (binary) positivity predicate $\ltimes$ which is dual to the cover $\lhd$. This brings a long standing problem to solution, namely the definition of a good predicative notion of formal closed subset.

All other notions of formal topology can be introduced by following the same method as described above. In this way one arrives at the notions of formal continuous relation and of formal map (see Section 2.4), and at that of formal point (see Section 2.4 and below). One could then say that formal topology is obtained by "forgetting" concrete points and thus by describing a concrete space (or more generally a basic pair) by using only what is available on its formal side, that of the set S. I hope it is clear by now that this is a colourful but approximate way of speaking, which can be useful for starting intuition. Though formal notions are introduced starting from concretely presentable ones, the aim remains that of reaching more general notions; if all formal (basic) topologies had happened to be concretely presentable, their introduction would have been much less motivated. The real *raison d'être* of formal topology is to include some topological structures which otherwise would be inaccessible to a predicative treatment.

Thus, as was the case with the rejection of excluded third, the formal approach should be perceived positively as an enrichment of topology, rather than negatively as a complicated way to mimick predicatively the results of impredicative topology. The notions of formal topology and of formal space are put *aside* that of concrete space, and they do not replace it. They are new conceptual tools, with their own results and techniques, like inductive generation, which often have no analogue in the pointwise approach. So in particular they are not just a skilful device to study topological spaces better, by simplifying proofs or other technical improvements. Also, they are not only a way to obtain more points, by the introduction of formal points (as discussed below). In many cases it happens that the collection of formal points over a formal topology can be indexed by a set; so the formal topology could possibly even become concretely presentable. But this is conceptually only a posteriori, and hence still the formal definition remains more natural and simple.

Formal topology provides us with another source of refinement of conceptual tools, and that is the distinction between concrete and formal points. The reason for introducing formal points is best understood by assuming the constructive viewpoint as an improvement on the quality of knowledge, rather than as doing without “strong” principles. Their aim is simply to increase expressive power while keeping constructivity. In fact, in the classical approach the most important examples of topological spaces are formed by points, like real numbers, infinitely proceeding sequences, etc., which contain, or are determined by, an infinite amount of information. Constructively, the only possibility is to conceive them as ideally determined by better and better approximations. One usually gives mathematical form to this idea by *defining* such ideal points as the collection of all the approximations. This is also the course taken in formal topology. In fact, a formal point of any formal topology $\mathcal{S}$ will be a subset α of the set S such that it makes sense to think of $a \,\varepsilon\, \alpha$ as meaning that the observable a is an approximation of α. To obtain a precise definition, one follows the same general method as described above for the definition of formal topologies. So one considers the case in which $\mathcal{S}$ is concretely presentable and takes the pointfree properties of the subset $\Diamond x$, which is the trace on S of a concrete point $x \in X$, as the conditions to define a subset $\alpha \subseteq S$ to be a formal point. The same conditions are then used to define formal points over an arbitrary formal topology (see Section 2.4). One could say that a subset α is a formal point if it enjoys enough properties to make it indistinguishable, in the presentable case, from (the trace of) a concrete point.

The collection of all formal points of $\mathcal{S}$ is denoted by $Pt(\mathcal{S})$, and it is called a formal space. From a constructive point of view, it is important to keep points which are given concretely, i.e. concrete points, well distinct from points which are only ideally so, i.e. formal points. It is the predicative foundation given by type theory, with its distinction between set and collection (or type), which allows one, and in the same time compels one to take care of this. In fact, each formal point is a subset of S, and hence $Pt(\mathcal{S})$ is a collection of subsets, which is *not* a set. In particular, quantifications over formal points are in general not meaningful. So the distinction between concrete points (when they exist) and formal points is simply that the former are elements and form a set, while the latter are subsets and do not. This distinction is lost in an impredicative approach, where $Pt(\mathcal{S})$ (or some other equivalent formulation) is considered to be a set as good as any other. So one can see that in a predicative approach formal points are not an option to reconstruct something which is there in any case, but a necessity to be able to deal with some spaces unreachable otherwise. In this precise sense, formal topology is predicatively not a luxury, but a must.

On the other hand, it should also be clear that in my opinion formal points are not a substitute to soothe the pain for the loss of concrete points. One should not use energy to avoid concrete points as evil, but to keep them distinct from the formal ones. The aim is just to develop topology predicatively, and also concrete points have shown to be useful in this respect.

The attitude described here can be traced back to 1873, when Dedekind introduced his rigorous explanation of the continuum. In his *Stetigkeit und irrationale Zahlen*, he analysed the effect of a concrete point, that is a rational number, on the set of approximations, and discovered that it gives what we now call a Dedekind cut. Then he reversed the perspective, and said that for any Dedekind cut one *creates* a new, ideal number. It was only later that Dedekind cuts, though infinite entities, were treated as concretely given objects. With the help of a predicative set theory, we can now refrain from that step. Formal topology is the general result growing out of this. So one could say that classical topology has been obtained from topology by "forgetting" that $Pt(\mathcal{S})$ is not a set, and hence also "forgetting" that half of topology which we now call pointfree or formal.

3.2.3. Deep connection between logic and topology

The basic picture shows that two sets linked by a relation are enough to give a beginning to topology, or at least to something very close to what is usually meant by topology. In fact, usual topology is obtained just by introducing the property of convergence of approximations, and this can be done quite easily by adding a condition on top of each definition.

On the other hand, the basic picture is in essence just the study of images of subsets along a relation, and such images are defined in terms of logical constants and quantifiers. Moreover, subsets themselves are just propositional functions. So one cannot be wrong to say that the basic picture is just applied logic. It comes from the dynamics between two sets which is induced by logic. That is precisely why I have called it the basic picture.

From the perspective of logic, the basic picture shows that topology is simply the study of combinations of quantifiers by means of definitions which put into action a spatial intuition. One could say that it gives a visual meaning to some combination of quantifiers.

From the perspective of topology, the basic picture offers a unifying structure which underlies the common topological definitions and thus a way to understand their deepest meaning. In this sense, it is a foundation of topology which is independent of any standard foundational theory.

In general, it seems safe to say that after the discovery of the basic picture the connection between logic and topology appears to be much deeper than it looked before.

3.2.4. Nothing wrong with axiomatic definitions

Some constructivists tend to believe constructive mathematics to be incompatible, perhaps even antithetic to the axiomatic method. This attitude is probably due to historical contingencies, bound to the formalistic foundation of mathematics as a purely linguistic game, but it has apparently no other reasonable motivation nowadays. An abstract, axiomatic definition can be useful in constructive mathematics in the same way as it is in classical mathematics (and which goes well beyond the formalistic view on foundations). First of all, it has the well-known and prosaic purpose of saving work, by finding those properties which hold in a variety of examples. In this way it can help the development of an abstract intuition, which may have a structural, logical or geometrical nature. This too might be a reason for the opposition of many constructivists who see, as Bishop did, the computational content as the only ingredient which should give meaning to mathematics. But this looks to me as irrational as forbidding oneself the use of knives because other people use them for bad purposes. In fact, one should trust in oneself and just avoid destructive misuse; the development of the axiomatic theory is fully independent of the way in which examples of it are conceived. These can (and must) remain as constructive as desired.

The lack of a tradition with axiomatic definitions in the field of constructive mathematics is not a good reason to take a classical definition for granted and just try to adapt it as well as possible to a constructive language. A reasonable criterion is that a good definition is the right compromise between convenience and faithfulness to reality, that is preservation of information. This is the criterion I followed in my choice of definitions in the specific case of the basic picture and formal topology. As a consequence, in the definition of formal topology I have not taken as axiom all what is valid in a concrete space (that is, a topological space to the eyes of classicists). The criterion of validity in a topological space is certainly important, but not the only one. There is no good reason to assume topological spaces as given in the same way as reality, transcending historical and human choices, and hence to conceive of formal topology as all what can be said about classical points without ever mentioning them. The definition of formal topology has its own status and autonomy (even if, of course, future understanding might change its present form).

More specifically, I have not taken the property of positivity (that is, $\mathsf{Pos}(a) \rightarrow a \lhd U / a \lhd U$, see Section 1.2 and footnote 13) as axiom. This allows to avoid problems with compactifications (see footnote 13); the compactification of a formal basic topology and of a formal topology (at least when it is provided with an operation $\cdot$ as in Definition 1) now gives no problems. Note also that if two formal topologies $\mathcal{S}$ and $\mathcal{S}'$ differ only in the sense that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by adding positivity as an axiom in the generation of the cover, then $\mathcal{S}$ and $\mathcal{S}'$ produce the same formal space. The most important advantage is however that in this way the definition is kept simple, general and (hopefully) deep.

For this same reason, I also believe that inductive generation should *not* be a part of the definition (this is also the choice taken in [10]). In fact, it can become a burden (also an impossible task, in some cases, see [10]) to prove a cover to be inductively generated when it is given by an elementary or an algebraic definition. Finally, avoiding the complexities of inductive generation in the abstract definition (one should always ask for the presence of an axiom set I, C, see [10] or also Section 2.6 here) looks to me as an example of good will in the effort of communication with the classical tradition.

3.2.5. Fun in doing constructive mathematics

One of the deep motivations pushing one into mathematical research is that one has fun in doing it. Working mathematicians know that fun is a sign for good mathematics. This is perhaps because fun is a measure of involvement and excitement for discoveries.

One of the deep reasons for little interest in constructive mathematics is, in my opinion, the prejudice that it must be terribly boring. Analysing how this prejudice was born is not easy, since it involves history, education, foundational attitude and even the global view of the world. It seems that it is due mainly to two reasons. One is that constructivity is often felt only as a moral duty, and this is often supposed to kill any fun. This is certainly due also to convinced constructivists themselves, who do not insist enough on the advantages, which are not only of moral nature, accompanying such a "duty", and hence which make it worthwhile. The other is simply the common, unjustified and condemnable ignorance about constructive mathematics. One just cannot have fun with something totally unfamiliar.

Explaining to some people why doing constructive mathematics can be fun, is just like explaining them why they should laugh for a certain joke. Terrible. What I can do is just on one hand to present myself as an example, and swear that I have real fun when I work, and on the other hand to warn that the kind of fun is a bit different from that experienced with classical mathematics. In fact, it is certainly true that working in constructive mathematics is conceptually more complex than in classical mathematics. Before having fun, one has to get a bit familiar with a foundation which is more complex than the classical one, and hence also to acquire a specific intuition. For a reader with a classical education, this means also abandoning some familiar schemes, and let oneself go into new mental structures. In return, one gets much more than in classical mathematics: computational meaning, safety, consistency, quality of information, contact with reality, etc. So the fun will be less infantile than in classical mathematics, since the exploration is deeper.

It has been a source of intense excitement for me to discover the beginning of the basic picture and to see how the correct definitions came out by purely structural reasons. Now there are several further topics which need exploration, which I expect to be great fun to do. Here are just a few examples: proof theoretic methods in topology (e.g. the problem in Section 1.3.6), co-induction, topology and mathematics with $\ltimes$ and $\between$ (see Section 2.7), different assumptions on $\ltimes$ and Pos, and their role, compactifications of formal (basic) topologies, etc.

I also expect many other interesting results to be found. In fact, using a new, constructive net does not mean throwing back to sea the treasures of mathematics so dear to Hilbert, but to the contrary it allows to acquire new treasures, which Hilbert could not see (or didn't want to).

3.2.6. Brouwer

Almost one century after Brouwer's beginning, it would be extremely interesting to read his writings again and carefully analyse his thought and his mathematics at the light of more recent developments of constructive mathematics, also in its connection with computers. A relaxed, less biased view should now be possible.

The difficulties of Brouwer's character (to which in the end all the difficulties in his thought can be reduced, like his tendency to solipsism and to polemics) should not prevent one from recognizing him as a pioneer and a prophet of exceptional depth. His insights, perhaps not all yet appreciated, began the creation of a whole new world and way of thinking.

The best way to continue his work is to follow his spirit, rather than his letter, in the development of mathematics. In particular, we are now free to look in a more relaxed way at some topics so dear to Brouwer in all his life, like choice sequences, continuity principles and bar induction, and which make intuitionistic mathematics incompatible with classical mathematics. They are certainly extremely interesting and subtle problems, but I believe not the essence of constructivity. Incompatibility remains, but it is more a matter of conceptual views than of specific mathematical principles. Still, a minimalist foundation, in which the axiom of choice is not valid and which is compatible with a direct intuition of continuity, could be the correct basis to address Brouwer's problems. In particular, an improvement on the attitude described in [32, Section 9], seems now possible. One of my main aims there was the representation of choice sequences as formal points. But one can show that, if the axiom of choice is assumed to hold, as I did, the presence of $\exists$ in the conclusion of the main condition of formal points brings to a function, so that the corresponding sequence is lawlike. I proposed in [32] to weaken that condition, by adding a double negation in front of $\exists$. (The suggestion was followed in [45] to obtain a formal space of "weak formal reals" which satisfies completeness.) This surely blocks the application of the axiom of choice, but vanishes the aim of identifying choice sequences with formal points. Now one can see the way out: the definition of formal point, being fully structural, is correct as it is. What is to be changed is the foundation: it is a foundation with no axiom of choice which allows one to identify choice sequences with formal points, since the argument showing that a formal point gives a lawlike sequence is no longer possible.

3.2.7. Bishop

In all his writings, Bishop insists on the idea of giving meaning to mathematics *only* through its computational content. He rightly does not identify mathematics with computer computation: "Because the computer is lacking in judgement, the theorems of constructive mathematics do not in general represent computer programs. They represent person programs, which in some instances can be transformed into computer programs and in other instances cannot." ([4, pp. 354–355]). However, in his writings he seems to ignore that mathematics is done by persons using also spatial intuition, or continuity, and abstract mental structures, or logic. Reducing everything to a single ingredient, like Pythagoreans, is no winning strategy. In particular, Bishop's opposition to general topology seems to me just as a mistake: "the flamboyant engine" of general topology has already "collapsed to constructive size" ([4, p. 63]), and that is through formal topology.

3.3. Pointless endless metareflections

The paper is now finally completed and, as with any piece of work in which I have been deeply involved, I deliver it with some trepidation. How will the readers receive it? After long doubts, I now believe it is pointless for me to suggest how they should. I believe that in the very end my duty towards life and evolution is just to preserve biodiversity in a cultural sense. So I first must exploit fully and personally the most precious organ I have as a human being, namely my brain, with no delegation to other brains, and then I must make the results available to others. That duty, at the moment, has been accomplished. As suggested by Bertolt Brecht (in his *Leben des Galilei*), after dreaming like Galilei, I now prefer to go back to the humble wisdom of Frau

Sarti:

> *Galilei (frühstückend): "Auf Grund unserer Forschungen, Frau Sarti, haben [... wir...] Entdekkungen gemacht, die wir nicht länger der Welt gegenüber geheimhalten können. Eine neue Zeit ist angebrochen, ein großes Zeitalter, in dem zu leben eine Lust ist."*
>
> *Frau Sarti: "So. Hoffentlich können wir auch den Milchmann bezahlen in dieser neuen Zeit, Herr Galilei."* [15]

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Taibaum (Eds.), Handbook of Logic in Computer Science, Vol. 3, Clarendon Press, Oxford, 1994, pp. 1–168.

[2] G. Battilotti, G. Sambin, A uniform presentation of sup-lattices, quantales and frames by means of infinitary preordered sets, pretopologies and formal topologies, Preprint No. 19, Dipartimento di Matematica, Università di Padova, November 1993.

[3] G. Battilotti, G. Sambin, Pretopologies and a uniform presentation of sup-lattices, quantales and frames, to appear.

[4] E. Bishop, Foundations of Constructive Analysis, McGraw-Hill, New York, 1967.

[5] E. Bishop, Schizophrenia in contemporary mathematics, in: M. Rosenblatt (Ed.), Errett Bishop: Reflections on Him and His Research, San Diego, CA, 1983, Contemporary Mathematics, Vol. 39, Amer. Math. Soc., Providence, RI, 1985, pp. 1–32.

[6] J. Cederquist, T. Coquand, Entailment relations and distributive lattices, in: Logic Colloquium '98, Lecture Notes in Logic, Vol. 13, Association of Symbolic Logic, Springer, Berlin, 2000, pp. 127–139.

[7] J. Cederquist, T. Coquand, S. Negri, The Hahn–Banach theorem in type theory, in: G. Sambin, J. Smith (Eds.), Twenty-Five Years of Constructive Type Theory, Proc. Congr. held in Venice, October 1995, Oxford University Press, Oxford, 1998, pp. 57–72.

[8] T. Coquand, An intuitionistic proof of Tychonoff's theorem, J. Symbolic Logic 57 (1992) 28–32.

[9] T. Coquand, S. Sadocco, G. Sambin, J. Smith, Formal topologies on the set of first-order formulae, J. Symbolic Logic 65 (2000) 1183–1192.

[10] T. Coquand, G. Sambin, J. Smith, S. Valentini, Inductively generated formal topologies, Ann. Pure Appl. Logic, to appear.

[11] G. Curi, Analisi costruttiva nella teoria dei tipi: topologie formali con diametro e distanza, Tesi di laurea in Matematica, Università di Padova, July 1998.

[12] G. Curi, Metric spaces in type theory via formal topology, in: D. Spreen (Ed.), Workshop on Domains IV, Rolandseck, Germany, 2-4 October 1998, Electronic Notes in Theoretical Computer Science, Vol. 35, Elsevier Science, Amsterdam, 2000.

[13] G. Curi, The points of (locally) compact regular formal topologies, in: P. Schuster, U. Berger, H. Osswald (Eds.), Reuniting the Antipodes—Constructive and Nonstandard Views of the Continuum, Symp. Proc., San Servolo, Venice, May 16-22, 1999, Synthese Library, Vol. 306, Kluwer, Dordrecht, 2002.

[14] G. Curi, Constructive metrizability in point-free topology, 2002, Theoret. Comput. Sci.

[15] R. Engelking, Outline of General Topology, North-Holland, Amsterdam, 1968.

[16] M.P. Fourman, R.J. Grayson, Formal Spaces, in: A.S. Troelstra, D. van Dalen (Eds.), The L.E.J. Brouwer Centenary Symp, North-Holland, Amsterdam, 1982, pp. 107–122.

[15] *Galilei (while having breakfast): "On the base of our researches, Ms. Sarti, we have made discoveries which we can no longer keep hidden to the world. A new time has begun, a great era in which to live is a pleasure." Ms. Sarti: "Really. I hope we can also pay the bill of the milkman in such a new time, Mr. Galilei".* I thank Maria and Dieter Spreen for their help with the translation.

[17] S. Gebellato, G. Sambin, The essence of continuity (the Basic Picture, II), Preprint No. 27, Dipartimento di Matematica, Università di Padova, December 2001.
[18] S. Gebellato, G. Sambin, Pointfree Continuity and Convergence, (the Basic Picture, II), Vol. IV, 2001, first draft.
[19] S. Gebellato, G. Sambin, The basic picture as invariance under transfer along a relation, (the Basic Picture, II), Vol. V, 2002, first draft.
[20] J. Girard, Linear Logic, Theoret. Comput. Sci. 50 (1987) 1–102.
[21] P.T. Johnstone, Stone Spaces, Cambridge University Press, Cambridge, 1982.
[22] J.L. Kelley, General Topology, van Nostrand, New York, 1955.
[23] M.E. Maietti, About effective quotients in constructive type theory, in: T. Altenkirch, W. Naraschewski, B. Reus (Eds.), Types for Proofs and Programs. TYPES '98, Lecture Notes in Computer Science, Vol. 1657, Springer, Berlin, 1999, pp. 164–178.
[24] M.E. Maietti, S. Valentini, Can you add power-sets to Martin-Löf's intuitionistic set theory? Math. Logic Quart. 45 (1999) 521–532.
[25] P. Martin-Löf, Notes on Constructive Mathematics, Almqvist & Wiksell, Stockholm, 1970.
[26] P. Martin-Löf, Constructive Mathematics and Computer Programming, in: L. Cohen, J. Łoš, H. Pfeiffer, K. Podowski (Eds.), Logic, Methodology and Philosophy of Science, Vol. VI, North-Holland, Amsterdam, 1982, pp. 153–175.
[27] P. Martin-Löf, Intuitionistic type theory. Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980, Bibliopolis, Napoli, 1984.
[28] S. Negri, D. Soravia, The continuum as a formal space, Arch. Math. Logic 38 (1999) 423–447.
[29] S. Negri, S. Valentini, Tychonoff's theorem in the framework of formal topologies, J. Symbolic Logic 62 (1997) 1315–1332.
[30] B. Nordström, K. Petersson, J.M. Smith, Programming in Martin-Löf's Type Theory, An Introduction, Oxford University Press, Oxford, 1990.
[31] V. Pratt, Chu spaces and their interpretation as concurrent objects, in: J. van Leeuwen (Ed.), Computer Science Today: Recent Trends and Developments, Lecture Notes in Computer Science, Vol. 1000, Springer, Berlin, 1995, pp. 392–405.
[32] G. Sambin, Intuitionistic formal spaces—a first communication, in: D. Skordev (Ed.), Mathematical Logic and its Applications, Plenum Press, New York, 1987, pp. 187–204.
[33] G. Sambin, Pretopologies and completeness proofs, J. Symbolic Logic 60 (1995) 861–878.
[34] G. Sambin, Formal topology and domains, in: D. Spreen (Ed.), Workshop on Domains IV, Rolandseck, Germany, 2-4 October 1998, Electronic Notes in Theoretical Computer Science, Vol. 35, Elsevier Science, Amsterdam, 2000.
[36] G. Sambin, Steps towards a dynamic constructivism, in: P. Gärdenfors, K. Kijania-Placek, J. Wolenski (Eds.), Proc. XI Internat. Congr., Logic, Methodology and Philosophy of Science, Krakow, August 1999, Kluwer, Dortdrecht, 2002, pp. 261–284.
[35] G. Sambin, The Basic Picture, a structure for topology, (the Basic Picture, II), I, Preprint No. 26, Dipartimento di Matematica, Università di Padova, December 2001.
[37] G. Sambin, Basic topologies, formal topologies, formal spaces, (the Basic Picture, II), III, in preparation.
[38] G. Sambin, S. Gebellato, A preview of the basic picture: a new perspective on formal topology, in: T. Altenkirch, W. Naraschewski, B. Reus (Eds.), Types for Proofs and Programs. TYPES '98, Lecture Notes in Computer Science, Vol. 1657, Springer, Berlin, 1999, pp. 194–207.
[39] G. Sambin, S. Valentini, Building up a toolbox for Martin-Löf's type theory: subset theory, in: G. Sambin, J. Smith (Eds.), Twenty-Five Years of Constructive Type Theory, Proc. Congr., held in Venice, October 1995, Oxford University Press, Oxford, 1998, pp. 221–244.
[40] G. Sambin, S. Valentini, P. Virgili, Constructive domain theory as a branch of intuitionistic pointfree topology, Theoret. Comput. Sci. 159 (1996) 319–341.
[41] D.S. Scott, Domains for denotational semantics, in: M. Nielsen, E.M. Schmidt (Eds.), Automata, Languages and programming, Springer, Berlin, 1982, pp. 577–613.
[42] I. Sigstam, Formal spaces and their effective presentations, Arch. Math. Logic 34 (1995) 211–246.
[43] M.B. Smyth, Topology, in: S. Abramsky, D.M. Gabbay, T.S.E. Taibaum (Eds.), Handbook of Logic in Computer Science, Vol. 1, Clarendon Press, Oxford, 1992, pp. 641–761.

[44] B. Solito, Gli operatori di interno e di chiusura da un punto di vista costruttivo. Tesi di Laurea in Matematica, Università di Padova, October 1998.
[45] D. Soravia, Numeri reali e analisi costruttiva nel contesto delle topologie formali. Tesi di Laurea in Matematica, Università di Padova, July 1995.
[46] S. Valentini, The forget-restore principle: a paradigmatic example, in: G. Sambin, J. Smith (Eds.), Twenty-Five Years of Constructive Type Theory, Proc. Congr., held in Venice, October 1995, Oxford University Press, Oxford, 1998, pp. 275–283.
[47] S. Valentini, A cartesian closed category in Martin-Löf's intuitionistic type theory, Theoret. Comput. Sci. 290 (2002) 189–219.
[48] S. Vickers, Topology via Logic, in: Cambridge Tracts in Theoretical Computer Science, Vol. 5, Cambridge University Press, Cambridge, 1989.
[49] K. Weihrauch, U. Schreiber, Embedding metric spaces into cpo's, Theoret. Comput. Sci. 16 (1981) 5–24.